INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5010 - Combinatorics and Graph Theory
Problem Sheet 3 Autumn 2023

Problem 1. How many ways can a True-False test be answered if there are 10 questions?

Problem 2. How many committees of 5 students can be selected from a class of 25?

Problem 3. How many even 4 digit integers can be made from the digits $1, 3, 4, 5, 6, 7$ if:

(a) repetition is not allowed

(b) repetition is allowed

Problem 4. Let

$$X = \{(a_1, a_2, a_3) : 1 \leq a_1, a_2, a_3 \leq 10, a_1, a_2, a_3 \text{ distinct}\},$$

$$Y = \{A \subseteq \{1, 2, \ldots, 10\} : |A| = 3\}.$$

Define a function $f : X \longrightarrow Y$ by $f((a_1, a_2, a_3)) = \{a_1, a_2, a_3\}$.

(a) Find $|X|$.

(b) Find the number of tuples $(a_1, a_2, a_3) \in X$ such that $f((a_1, a_2, a_3)) = \{2, 3, 4\}$.

(c) Find $|Y|$.

Problem 5. Prove that

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

for $n, r \in \mathbb{N}$ in two ways: using algebraic manipulation; bijection with some of the subsets of $\{1, \ldots, n\}$.

Problem 6. Prove that

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

for $n, r \in \mathbb{N}$ in two ways: using induction; bijection with some of the subsets of $\{1, \ldots, n+1\}$.

Problem 7. Find the number of 5-tuples of the form $(x_1, x_2, x_3, x_4, x_5)$, where each x_i is a nonnegative integer and $x_1 + x_2 + x_3 + x_4 + 3x_5 = 12$.

Problem 8. After expanding $(a + b + c + d)^{2022}$ and combining like terms, how many terms are there?

Problem 9. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection (Try to give at least two proofs: one using pigeon-hole principle and other without using pigeon-hole principle).

Problem 10. Given any $n + 1$ distinct integers between 1 and $2n$, show that two of them are relatively prime. Is this result best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

Problem 11. Prove that among 502 positive integers, there are always two integers so that either their sum or their difference is divisible by 1000.

Problem 12. The sum of one hundred given real number is zero. Prove that at least 99 of the pairwise sums of these hundred numbers are nonnegative. Is this result the best possible one?