# INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

# MA5010/MA1240 - Combinatorics and Graph Theory/Combinatorics

## Problem Sheet 1 Autumn 2023

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**Problem 1.** Do there exist graphs with the following degree sequences:

(a) $2, 3, 4, 4, 5$.

(b) $0, 1, 2, 3, 4, 5, 6$.

(c) $1, 1, 1, 1, 1, 1$

(d) $1, 2, 3, 4, 5, 6$

**Problem 2.** Prove that if a graph $G$ has at least two vertices, then $G$ contains two vertices of the same degree.

**Problem 3.** Prove that if there is a walk in the graph $G$ between the vertices $u$ and $v$, then there is a path between the vertices $u$ and $v$ in $G$. Also, show that every circuit contains a cycle.

**Problem 4.** Let $G$ be a graph and $G^c$ be the complement of $G$. Show that either $G$ or $G^c$ is connected.

**Problem 5.** Show that in every connected graph $|V(G)| > 2$ there exists a vertex so that $G \setminus v$ is connected.

**Problem 6.** Let $G$ be a tree and $v$ be a vertex in $G$. Then all the connected components of the graph $G - v$, the graph obtained from $G$ by deleting $v$, are trees. These connected components are called the branches at the vertex $v$. Show that every tree contains a vertex such that every branch at this vertex contains at most half the vertices of the tree.

**Problem 7.** Give an example of a connected graph containing more cut-edges (bridges) than the cut-vertices

**Problem 8.** Let $G$ be a connected graph on $n$ vertices other than the complete graph. If $e$ is a bridge of $G$, then show that $e$ is incident with a cut-vertex in $G$.

**Problem 9.** Let $G$ be a connected graph containing only even vertices. Prove that $G$ cannot contain cut-edges.

**Problem 10.** If a tree $G$ contains a vertex of degree $d$, then $G$ contains at least $d$ pendent vertices.

**Problem 11.** Let $G$ be a tree. Show that any two maximum-paths in $G$ must have a common vertex.

**Problem 12.** Let $n \geq 2$. Show that there is a tree with degree sequence $(d_1, d_2, \ldots, d_n)$ if and only if $d_i > 0$ for all $i$ and $\sum_{i=1}^{n} d_i = 2(n-1)$.

**Problem 13.** Let $G$ be a weighted graph. Consider the following algorithm to construct a minimum spanning tree: Choose $v_1$ in $G$. Choose an edge incident with $v_1$ with minimum weight. After picking $S = \{v_1, \ldots, v_k\}$, choose an edge with one endpoint in $S$ and another in $S^c$ and with the smallest weight among all such edges. Let $v_{k+1}$ be the endpoint of this edge not in $S$, and add this vertex and the associated edge to $T$. Continue until all vertices of $G$ are in $T$. Show that any tree obtained by the above algorithm is a minimum spanning tree.

**Problem 14.** Compute the number of spanning trees in the following graphs:

(a) Complete graph on $n$ vertices minus one edge.

(b) The complete bipartite graph $K_{p,q}$.