Adjacency matrices of graphs

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Graph

- A graph G is a pair (V, E), where $V = \{v_1, \dots, v_n\}$ is the vertex set of G, and $E = \{e_1, \dots, e_m\} \subseteq V \times V$ is the edge set G.

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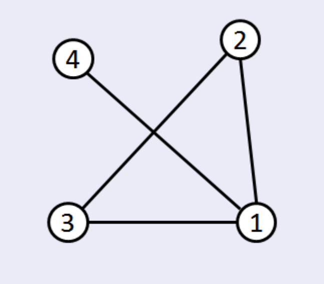

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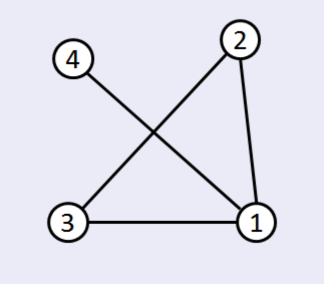

- We will work with simple, finite, undirected graphs.

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- A component is maximal connected subgraph of a graph G.

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- The diameter of a graph G is $diam(G) = \max\{d(v_i, v_j)\}$

Adjacency matrix

Definition (Adjacency matrix)

The adjacency matrix of a graph G with n vertices, $V(G) = \{v_1, ... v_n\}$ is a $n \times n$ matrix, denoted by $A(G) = (a_{ij})$, and is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Example

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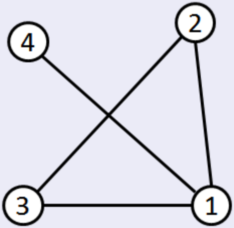
Consider the graph G



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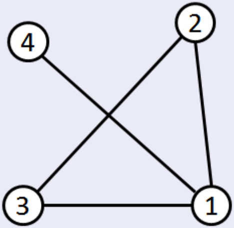

The **adjacency matrix** of G is

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Properties

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6. Let G be a connected graph with k distinct eigenvalues and let d be the diameter of G. Then $k > d$.

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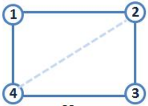
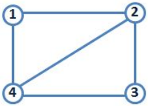
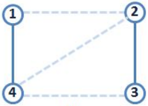


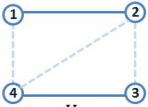
H_1



G



H_2



H_3

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- For $H \in \mathcal{H}(G)$, let $c(H)$ and $c_1(H)$ denotes the number of components in H which are cycles and edges, respectively.

Determinant of adjacency matrix

Theorem

Let G be a simple graph with $V(G) = \{v_1, v_2, ..., v_n\}$ and A be its adjacency matrix. Then,

$$\det(A) = \sum (-1)^{n - c_1(H) - c(H)} 2^{c(H)},$$

where the summation is over all spanning elementary subgraphs H of G.

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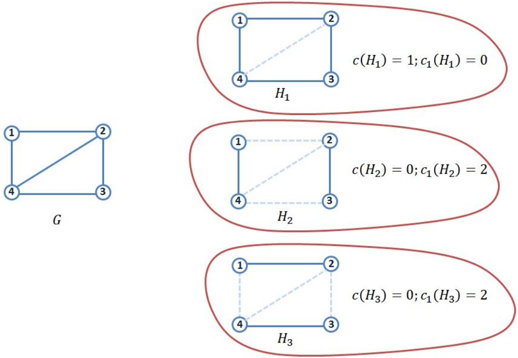

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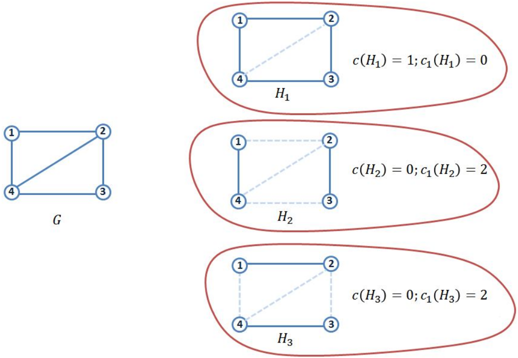

$$\det(A) = (-1)^{4-0-1}2^1 + (-1)^{4-2} + (-1)^{4-2} = 0$$

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Theorem

Let T be a tree with $V(T) = \{1, 2, .., n\}$, and let A be the adjacency matrix of T. Then, A is nonsingular if and only if T has a perfect matching.

Graph theoretical interpretation of inverse of a tree

Definition

Let T be a tree with a perfect matching $\mathcal{M}$. A path $P(v_i, v_j)$ is called an alternating path, if the edges are alternately in $\mathcal{M}$ and $\mathcal{M}^c$, and the first and last edges are in $\mathcal{M}$.

Theorem

Let T be a nonsingular tree with $V(T) = \{1, 2, ..., n\}$ and let A be the adjacency matrix of T. Let $\mathcal{M}$ be the perfect matching of T. Let $B = [b_{ij}]$ be the $n \times n$ matrix define as follows: $b_{ij} = 0$ if $i = j$ or if $P(i, j)$ is not alternating. If $P(i, j)$ is alternating, then set

$$b_{ij} = (-1)^{\frac{d(i,j)-1}{2}}.$$

Then $B = A^{-1}$.

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To join the group on "Graphs, matrices and their applications" mail to:
rajeshkannan1.m@gmail.com

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