INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
MA60053 - Computational Linear Algebra
Problem Sheet - Eigenvalue problems
Spring 2020

Problem 1 Let $A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{n\times m}$. Show that the matrices

$$\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

are similar. Using this, show that the nonzero eigenvalues of the matrices AB and BA are the same.

Problem 2 Let $A \in \mathbb{R}^{n\times n}$, and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Show that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \min_{\det(S)\neq 0} \|S^{-1}AS\|_F^2. \tag{1}$$

If A is normal, then show that equality holds in (1).

Problem 3 Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively. Let $\nu_1 \geq \cdots \geq \nu_n$ be the eigenvalues of $A + B$. Show that $\lambda_j + \mu_n \leq \nu_j \leq \lambda_j + \mu_1$ for all $1 \leq j \leq n$.

Problem 4 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Show that, for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n \setminus \{0\}$,

$$\min_{1\leq j\leq n} |\lambda - \lambda_j| \leq \frac{\|\lambda x - Ax\|_2}{\|x\|_2}.$$

Problem 5 Let $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, $\|x\|_2 = 1$ and λ be a scalar. If $r = Ax - \lambda x$, then, show that, there exists a rank-one matrix E such that $\|E\|_F = \|r\|_2$ and $(A + E)x = \mu x$.

Problem 6 Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively. Then,

$$|\lambda_j - \mu_j| \leq \|A - B\|,$$

for every compatible matrix norm.

Problem 7 Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$. Let $\sigma_1 \geq \cdots \geq \sigma_n$ be the singular values of A. Show that

$$\sigma_j = \min_{U\in\mathcal{M}_{n+1-j}} \max_{x\in U, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}, \qquad 1 \leq j \leq n,$$

where $\mathcal{M}_j$ is the set of all j-dimensional subspaces of $\mathbb{R}^n$.

A matrix $A \in \mathbb{R}^{n\times n}$ is upper Heisenberg, if $a_{ij} = 0$, for $j = 1, \ldots, n-2, \ \ i = j+2, \ldots, n$. The matrix is reduced if and only if $A_{i+1,i} \neq 0$ for $i = 1, \ldots, n-1$.

Problem 8 Let H be an upper Heisenberg matrix. Define $H_1 = H$ and $Q = I$. For $k \geq 1$, $H_k = Q_{k+1}R_{k+1}$(QR-factorization); $H_{k+1} = R_{k+1}Q_{k+1}$. Show that the matrices H_k are orthongonally similar for $k \geq 1$, and they are upper Heisenberg too.