# INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
## MA60053 - Computational Linear Algebra
## Problem Sheet - Singular Value Decomposition
## Spring 2020

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**Problem 1** Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. Show that the singular values of the matrix

$$\begin{bmatrix} I_n \\ A \end{bmatrix}$$

are equal to $\sqrt{1+\sigma_j^2}$, where $1 \leq j \leq n$.

**Problem 2** Let $A, B \in \mathbb{C}^{m\times n}$. Show that the matrices $A$ and $B$ have the same singular values if and only if there exists unitary matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that $B = PAQ$.

**Problem 3** Let $A \in \mathbb{C}^{k\times m}, B \in \mathbb{C}^{m\times n}$, $q = \min\{k, n\}$, and $p = \min\{m, n\}$. Show that

$$\sigma_1(AB) \leq \sigma_1(A)\sigma_1(B) \quad \text{and} \quad \sigma_q(AB) \leq \sigma_1(A)\sigma_p(B).$$

**Problem 4** Let $A \in \mathbb{C}^{m\times n}$ with $m > n$, $z \in \mathbb{C}^m$, and

$$B = \begin{bmatrix} A & z \end{bmatrix}.$$

Show that $\sigma_{n+1}(B) \leq \sigma_n(A)$ and $\sigma_1(B) \geq \sigma_1(A)$.

**Problem 5** Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, $z \in \mathbb{C}^n$, and

$$B = \begin{bmatrix} A \\ z^* \end{bmatrix}.$$

Show that $\sigma_n(B) \geq \sigma_n(A)$ and $\sigma_1(A) \leq \sigma_1(B) \leq \sqrt{\sigma_1^2(A)) + \|z\|_2^2}$

**Problem 6** Let $A \in \mathbb{C}^{n\times n}$ be nilpotent such that $A^j = 0$ and $A^{j-1} \neq 0$ for some $j \geq 1$. Let $b \in \mathbb{C}^n$ with $A^{j-1}b \neq 0$. Show that

$$K = \begin{bmatrix} b & Ab & \dots & A^{j-1}b \end{bmatrix}$$

has full column rank.

**Problem 7** Let $A \in \mathbb{C}^{n\times n}$. Show that there exists a unitary matrix $Q$ such that $A^* = QAQ$.

**Problem 8** Let $A \in \mathbb{C}^{m\times n}$ has rank $n$. Show that there is a factorization of $A = PH$, where $P \in \mathbb{C}^{m\times n}$ has orthonormal columns, and $H \in \mathbb{C}^{n\times n}$ is Hermitian positive definite. If $A \in \mathbb{C}^{n\times n}$, then, show that, $\|A - P\|_2 \leq \|A - Q\|_2$ for any unitary matrix $Q$.

**Problem 9** Let $A \in \mathbb{C}^{m\times n}$ has rank $n$ have a factorization $A = PH$ as in Problem 8. Show that

$$\frac{\|A^*A - I\|_2}{1 + \|A\|_2} \leq \|A - P\|_2 \leq \frac{\|A^*A - I\|_2}{1 + \sigma_n}$$

**Problem 10** Let $A \in \mathbb{C}^{n\times n}$ and $\sigma > 0$. Show that $\sigma$ is a singular values of $A$ if and only if the matrix

$$\begin{bmatrix} A & -\sigma I \\ -\sigma I & A^* \end{bmatrix}$$

is singular.