INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
MA60053 - Computational Linear Algebra
Problem Sheet - Iterative methods for linear systems
Spring 2020

Problem 1 If the matrix A is symmetric positive definite, then, show that, the Gauss-Seidel method converges. What about the convergence of Jacobi method?

A matrix A is *strictly diagonally dominant* if

$$\sum_{j=1 j\neq i}^{n} |a_{ij}| < |a_{ii}| \text{ for all } i = 1, \dots, n.$$

Consider the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

Problem 2 If the matrix A is strictly diagonally dominant, then show that the Jacobi method and Gauss-Seidel method converges. Does every strictly diagonally dominant A necessarily positive definite?

Problem 3 For a symmetric positive definite matrix A, the relaxation method converges if and only if $0 < \omega < 2$.

Problem 4 Let A be an $n \times n$ complex matrix with $\rho(A) < 1$. Then, show that the matrix $(I - M)$ is invertible, and $(I - M)^{-1} = I + M + M^2 + \dots$, the series on the right converging. Conversely, if the series on the right converges, then $\rho(A) < 1$.

Consider the linear system $Ax = b$ where A is an $n \times n$ matrix, and let $A = M - N$ be a splitting of A, where M is invertible. Consider the iterative scheme associated with this splitting: $x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b$.

Problem 5 Show that the iterative method given by the splitting generalizes the Jacobi method, the Gauss-Seidel method and the successive over relaxation method, by provinding proper choices of M and N.

A matrix $A \in \mathbb{C}^{n\times n}$ is *convergent* if $\rho(A) < 1$.

Problem 6 Let $A \in \mathbb{C}^{n\times n}$. Show that the following are equivalent:

1. A is convergent.
2. $\lim_{n\to\infty} A^k = 0$.
3. $\lim_{n\to\infty} A^k x = 0, \text{ for all } x \in \mathbb{C}^n$.
4. $\|B\| < 1$, for at least one matrix norm.

Problem 7 Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. The solution of the system $Ax = b$ is the unique minimum of the function $f(y) = \frac{1}{2}y^T Ay - y^T b$.

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$, the *Krylov subspaces* are defined as follow:

$$\mathcal{K}_k(b, A) = \text{span}\{b, Ab, \dots, A^{k-1}b\}, \quad k = 1, 2 \dots.$$

Problem 8 If the minimal polynomial of a nonsingular matrix A has degree m, then, show that, the solution to $Ax = b$ lies in the subspace $\mathcal{K}_m(b, A)$.