MA60053 - Computational Linear Algebra
Problem Sheet 3

Problem 1. *Let A be a symmetric positive definite matrix. Two vectors u and v are said to be A-orthogonal if $u_1^T A u_2 = 0$. Show that every subspace has an A-orthonormal basis.*

Problem 2. *Let $x \in \mathbb{R}^n$ and let P be a Householder matrix such that $Px = \pm||x||_2 e_1$. Let $G_{1,2}, \ldots, G_{n-1,n}$ be Givens rotations, and let $Q = G_{1,2} \ldots G_{n-1,n}$. Suppose $Qx = \pm||x||_2 e_1$. Must P equals to Q?*

Problem 3. *Let A be an $n \times m$ real matrix. Show that $X = A^\dagger$ minimizes $||AX - I||_F$ over all $m \times n$ matrices X. What is the value of this minimum?*

Problem 4. *Let $H = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$, where $A = U\Sigma V^T$ is the SVD of an $n \times n$ matrix A. Let $\Sigma = diag(\sigma_1, \ldots, \sigma_n), U = [u_1, \ldots, u_n]$ and $V = [v_1, \ldots, v_n]$. Then the $2n$ eigenvalues of H are $\pm\sigma_i$, with corresponding unit eigenvectors $\frac{1}{\sqrt{(2)}} \begin{pmatrix} v_i \\ \pm u_i \end{pmatrix}$.*

Problem 5. *Let $B \in \mathbb{R}^{n\times m}$ be any matrix such thta $R(A) = R(B)$. Show that x is a solution of the least squares problem for the overdetermined system $Ax = b$ if and only if $B^T Ax = B^T x$.*

Problem 6. *Show that if A has full rank, then $R(A) = R(B)$ if and only if there exists a nonsingular matrix $C \in \mathbb{R}^{m\times m}$ such that $A = BC$. What happens if we drop the assumption that A has full rank.*

Problem 7. *Show that the function $f(x_1, \ldots, x_n) = ||b - Ax||_2^2$ is a differentiable function on m variables. Compute $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m})^T$ and, using this, derive the normal equation for the least squares problem.*

Problem 8. *Let $A \in \mathbb{R}^{n\times m}$ with singular values $\sigma_1 \geq \ldots, \geq \sigma_m$ and right singular vectors $v_1, \ldots, v_m$. Show that for $k = 1, \ldots, m$, $\sigma_k = \max\{\frac{||Ax||_2}{||x||_2} : x \neq 0, x \in span\{v_1, v_2, \ldots, v_{k-1}\}^\perp\} = \min\{\frac{||Ax||_2}{||x||_2} : x \neq 0, x \in span\{v_{k+1}, \ldots, v_m\}^\perp\}$.*

Problem 9. *Let $A \in \mathbb{R}^{n\times m}$. Then B is the Pseudoinverse of A if and only if B satisfies the following four equations(Moore-Penrose equations):*

1. $ABA = A$
2. $BAB = B$
3. $(BA)^T = BA$
4. $(AB)^T = AB$.