Adjacency matrices of complex unit gain graphs

M. Rajesh Kannan

Department of Mathematics,
Indian Institute of Technology Kharagpur,
email: rajeshkannan1.m@gmail.com, rajeshkannan@maths.iitkgp.ac.in

February 13, 2019

Outline

- Adjacency matrices of graphs
- Perron-Frobenius theorem
- Spectral properties
- Adjacency matrices of complex unit gain graphs
- Characterization of bipartite graphs and trees

Adjacency matrix

Definition (Adjacency matrix)

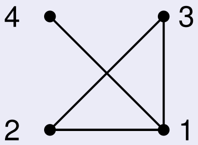
The adjacency matrix of a graph G with n vertices, $V(G) = \{v_1, ...v_n\}$ is a $n \times n$ matrix, denoted by $A(G) = (a_{ij})$, and is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Example

Example

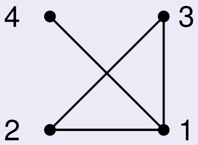
Consider the graph G



Example

Example

Consider the graph G



The **adjacency matrix** of G is

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Properties

Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.

Properties

Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.

Properties

Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.
3. Sum of the 3×3 principal minors of A equals to twice the number of triangles in the graph.

Properties

Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.
3. Sum of the 3×3 principal minors of A equals to twice the number of triangles in the graph.
4. $(i, j)^{th}$ entry of the matrix A^k equals the number of walks of length k from the vertex i to the vertex j.

Properties

Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.
3. Sum of the 3×3 principal minors of A equals to twice the number of triangles in the graph.
4. $(i,j)^{th}$ entry of the matrix A^k equals the number of walks of length k from the vertex i to the vertex j.
5. If v_i and v_j are vertices of G with $d(v_i, v_j) = m$, then the matrices $I, A, \ldots, A^m$ are linearly independent.

Properties

Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.
3. Sum of the 3×3 principal minors of A equals to twice the number of triangles in the graph.
4. $(i, j)^{th}$ entry of the matrix A^k equals the number of walks of length k from the vertex i to the vertex j.
5. If v_i and v_j are vertices of G with $d(v_i, v_j) = m$, then the matrices $I, A, \ldots, A^m$ are linearly independent.

Irreducible matrices

An $n \times n$ matrix, $n \geq 2$, is *reducible* its rows and columns can be simultaneously permuted to

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square (not necessarily of the same order).

Irreducible matrices

An $n \times n$ matrix, $n \geq 2$, is *reducible* its rows and columns can be simultaneously permuted to

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square (not necessarily of the same order). Otherwise, it is *irreducible*. For $n = 1$, 0 is reducible, $a \neq 0$ is irreducible.

Irreducible matrices

An $n \times n$ matrix, $n \geq 2$, is *reducible* its rows and columns can be simultaneously permuted to

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square (not necessarily of the same order). Otherwise, it is *irreducible*. For $n = 1$, 0 is reducible, $a \neq 0$ is irreducible.

The *directed graph* $G(A)$, associated with an $n \times n$ matrix has n vertices $1, \ldots, n$ and an arc from i to j if and only if $a_{ij} \neq 0$.

Irreducible matrices

An $n \times n$ matrix, $n \geq 2$, is *reducible* its rows and columns can be simultaneously permuted to

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square (not necessarily of the same order). Otherwise, it is *irreducible*. For $n = 1$, 0 is reducible, $a \neq 0$ is irreducible.

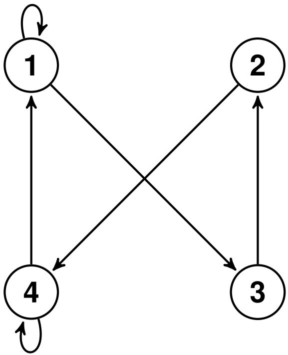
The *directed graph* $G(A)$, associated with an $n \times n$ matrix has n vertices $1, \ldots, n$ and an arc from i to j if and only if $a_{ij} \neq 0$.

Working definition: A is irreducible if and only if $G(A)$ is strongly connected.

Example

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Example

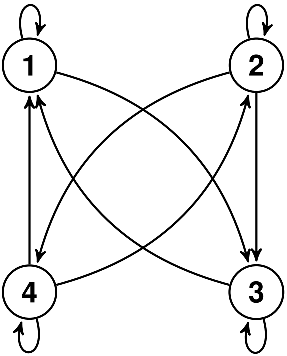


$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Example



$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Perron-Frobenius Theorem

Theorem

If A is nonnegative and irreducible, then

a) $\rho(A) > 0$, *where* $\rho(A)$ *is the maximum of absolute value of all the eigenvalues of* A,
b) $\rho(A)$ *is an eigenvalue of* A,
c) *There is a positive vector such that* $Ax = \rho(A)x$,

Perron-Frobenius Theorem

Theorem

If A is nonnegative and irreducible, then

a) $\rho(A) > 0$, *where* $\rho(A)$ *is the maximum of absolute value of all the eigenvalues of* A*,*
b) $\rho(A)$ *is an eigenvalue of* A*,*
c) *There is a positive vector such that* $Ax = \rho(A)x$*,*

Theorem

Let $A, B \in \mathbb{C}^{n\times n}$ *and suppose that* A *is nonnegative. If* $A \geq |B|$*, then* $\rho(A) \geq \rho(|B|) \geq \rho(B)$.

Perron-Frobenius Theorem

Theorem

If A is nonnegative and irreducible, then

a) $\rho(A) > 0$, *where* $\rho(A)$ *is the maximum of absolute value of all the eigenvalues of* A,
b) $\rho(A)$ *is an eigenvalue of* A,
c) *There is a positive vector such that* $Ax = \rho(A)x$,

Theorem

Let $A, B \in \mathbb{C}^{n\times n}$ *and suppose that* A *is nonnegative. If* $A \geq |B|$, *then* $\rho(A) \geq \rho(|B|) \geq \rho(B)$.

Theorem

Let $A, B \in \mathbb{C}^{n\times n}$. *Suppose* A *is nonnegative and irreducible, and* $A \geq |B|$. *If* $\lambda = e^{i\theta}\rho(B)$ *is a maximum-modulus eigenvalue of* B,

Perron-Frobenius Theorem

Theorem

If A is nonnegative and irreducible, then

a) $\rho(A) > 0$, *where* $\rho(A)$ *is the maximum of absolute value of all the eigenvalues of* A*,*
b) $\rho(A)$ *is an eigenvalue of* A*,*
c) *There is a positive vector such that* $Ax = \rho(A)x$*,*

Theorem

Let $A, B \in \mathbb{C}^{n\times n}$ *and suppose that* A *is nonnegative. If* $A \geq |B|$*, then* $\rho(A) \geq \rho(|B|) \geq \rho(B)$.

Theorem

Let $A, B \in \mathbb{C}^{n\times n}$*. Suppose* A *is nonnegative and irreducible, and* $A \geq |B|$*. If* $\lambda = e^{i\theta}\rho(B)$ *is a maximum-modulus eigenvalue of* B*, then there is a diagonal unitary matrix* $D \in \mathbb{C}^{n\times n}$ *such that* $B = e^{i\theta}DAD^{-1}$.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

Properties of spectrum

- $\delta(G) \leq \lambda_1 \leq \Delta(G)$.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

Properties of spectrum

- $\delta(G) \leq \lambda_1 \leq \Delta(G)$.
- $\chi(G) \leq 1 + \lambda_1$, where $\chi(G)$ is the chromatic number of G.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

Properties of spectrum

- $\delta(G) \leq \lambda_1 \leq \Delta(G)$.
- $\chi(G) \leq 1 + \lambda_1$, where $\chi(G)$ is the chromatic number of G.
- $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

Properties of spectrum

- $\delta(G) \leq \lambda_1 \leq \Delta(G)$.
- $\chi(G) \leq 1 + \lambda_1$, where $\chi(G)$ is the chromatic number of G.
- $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$.
- G is bipartite if and only if the eigenvalues of A are symmetric with respect to origin.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

Properties of spectrum

- $\delta(G) \leq \lambda_1 \leq \Delta(G)$.
- $\chi(G) \leq 1 + \lambda_1$, where $\chi(G)$ is the chromatic number of G.
- $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$.
- G is bipartite if and only if the eigenvalues of A are symmetric with respect to origin. That is, λ is an eigenvalue of $A(G)$ if and only if $-\lambda$ is an eigenvalue of $A(G)$.

Gain graphs

- Let $\mathfrak{G}$ be a group and, let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.

Gain graphs

- Let $\mathfrak{G}$ be a group and, let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.
- Define e_{jk} as a directed edge from the vertex j to the vertex k, if there is an edge between them.

Gain graphs

- Let $\mathfrak{G}$ be a group and, let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.
- Define e_{jk} as a directed edge from the vertex j to the vertex k, if there is an edge between them.
- The directed edge set $\overrightarrow{E(G)}$ consists of the directed edges $e_{jk}, e_{kj} \in \overrightarrow{E(G)}$, for each adjacent vertices j and k of G.

Gain graphs

- Let $\mathfrak{G}$ be a group and, let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.
- Define e_{jk} as a directed edge from the vertex j to the vertex k, if there is an edge between them.
- The directed edge set $\overrightarrow{E(G)}$ consists of the directed edges $e_{jk}, e_{kj} \in \overrightarrow{E(G)}$, for each adjacent vertices j and k of G.
- Assign a weight (gain) $g \in \mathfrak{G}$ for each directed edge $e_{jk} \in \overrightarrow{E(G)}$, such that the weight of e_{kj} is g^{-1}. Let us denote this assignment by φ.

$\mathbb{T}$-gain adjacency matrix

Definition (Thomas Zaslavsky)

A $\mathfrak{G}$-gain graph is a graph G in which each orientation of an edge is given a gain which is the inverse of the gain assigned to the opposite orientation.

$\mathbb{T}$-gain adjacency matrix

Definition (Thomas Zaslavsky)

A $\mathfrak{G}$-gain graph is a graph G in which each orientation of an edge is given a gain which is the inverse of the gain assigned to the opposite orientation.

If $\mathfrak{G} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then the gain graph is called $\mathbb{T}$-gain graph.

$\mathbb{T}$-gain adjacency matrix

Definition (Thomas Zaslavsky)

A $\mathfrak{G}$-gain graph is a graph G in which each orientation of an edge is given a gain which is the inverse of the gain assigned to the opposite orientation.

If $\mathfrak{G} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then the gain graph is called $\mathbb{T}$-gain graph.

Definition ($\mathbb{T}$-gain adjacency matrix)

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$- gain graph, where $\varphi : \overrightarrow{E(G)} \to \mathbb{T}$ be a weight function.

$\mathbb{T}$-gain adjacency matrix

Definition (Thomas Zaslavsky)

A $\mathfrak{G}$-gain graph is a graph G in which each orientation of an edge is given a gain which is the inverse of the gain assigned to the opposite orientation.

If $\mathfrak{G} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then the gain graph is called $\mathbb{T}$-gain graph.

Definition ($\mathbb{T}$-gain adjacency matrix)

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$- gain graph, where $\varphi : \overrightarrow{E(G)} \to \mathbb{T}$ be a weight function. The $\mathbb{T}$**-gain adjacency matrix** or complex unit gain adjacency matrix $A(\Phi) = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

On $\mathbb{T}$-gain adjacency matrix

Example

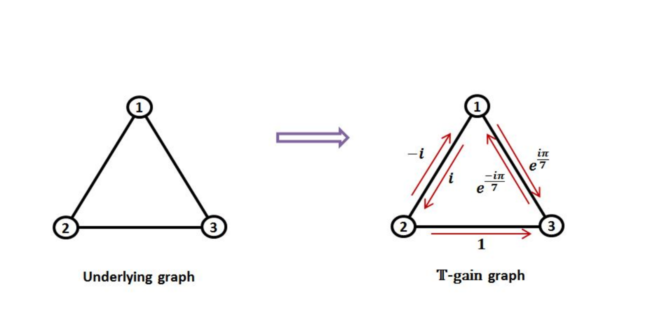


Figure: $\mathbb{T}$-gain graph Φ and its underlying graph

On $\mathbb{T}$-gain adjacency matrix

Example

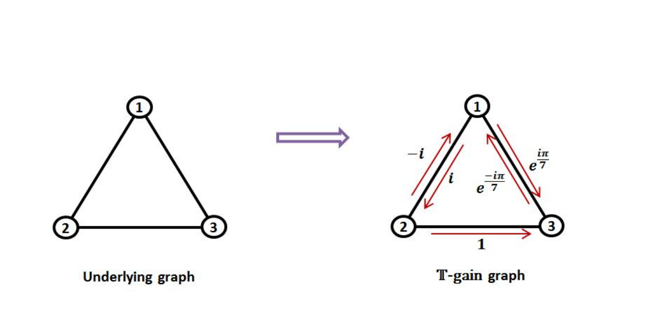


Figure: $\mathbb{T}$-gain graph Φ and its underlying graph

$$A(\Phi) = \begin{pmatrix} 0 & i & e^{i\frac{\pi}{7}} \\ -i & 0 & 1 \\ e^{-i\frac{\pi}{7}} & 1 & 0 \end{pmatrix}$$

Definition

- **The gain of a cycle** $C = v_1 v_2, \ldots v_l v_1$, denoted by $\varphi(C)$, is defined as the product of the gains of its edges, that is $\varphi(C) = \varphi(e_{12})\varphi(e_{23})\ldots\varphi(e_{(l-1)l})\varphi(e_{l1})$.

Definition

- **The gain of a cycle** $C = v_1 v_2, \dots v_l v_1$, denoted by $\varphi(C)$, is defined as the product of the gains of its edges, that is $\varphi(C) = \varphi(e_{12})\varphi(e_{23})\dots\varphi(e_{(l-1)l})\varphi(e_{l1})$.
- A cycle C is said to be **neutral** if $\varphi(C) = 1$, and a gain graph is said to be **balanced** if all its cycles are neutral.

Definition

- **The gain of a cycle** $C = v_1 v_2, \dots v_l v_1$, denoted by $\varphi(C)$, is defined as the product of the gains of its edges, that is $\varphi(C) = \varphi(e_{12})\varphi(e_{23}) \dots \varphi(e_{(l-1)l})\varphi(e_{l1})$.
- A cycle C is said to be **neutral** if $\varphi(C) = 1$, and a gain graph is said to be **balanced** if all its cycles are neutral.
- A function from the vertex set of G to the complex unit circle $\mathbb{T}$ is called a **switching function**.

Definition

- **The gain of a cycle** $C = v_1 v_2, \ldots v_l v_1$, denoted by $\varphi(C)$, is defined as the product of the gains of its edges, that is $\varphi(C) = \varphi(e_{12})\varphi(e_{23})\ldots\varphi(e_{(l-1)l})\varphi(e_{l1})$.
- A cycle C is said to be **neutral** if $\varphi(C) = 1$, and a gain graph is said to be **balanced** if all its cycles are neutral.
- A function from the vertex set of G to the complex unit circle $\mathbb{T}$ is called a **switching function**.
- We say that, two gain graphs $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ are said to be **switching equivalent, written as** $\Phi_1 \sim \Phi_2$, if there is a switching function $\zeta : V \to \mathbb{T}$ such that $\varphi_2(e_{ij}) = \zeta(v_i)^{-1}\varphi_1(e_{ij})\zeta(v_j)$.

Spectrum of $\mathbb{T}$-gain adjacency matrix

Theorem (Zaslavsky[19],1989)

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. Then Φ is balanced if and only if $\Phi \sim (G, 1)$.

Spectrum of $\mathbb{T}$-gain adjacency matrix

Theorem (Zaslavsky[19],1989)

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. Then Φ is balanced if and only if $\Phi \sim (G, 1)$.

Theorem (Reff[17], 2012)

Let $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ be two $\mathbb{T}$-gain graph. If $\Phi_1 \sim \Phi_2 \Rightarrow A(\Phi_1)$ and $A(\Phi_2)$ have the same spectrum.

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ *be a* $\mathbb{T}$*-gain (connected) graph, then* $\rho(A(\Phi)) = \rho(A(G))$ *if and only if either* Φ *or* $-\Phi$ *is balanced.*

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either Φ or $-\Phi$ is balanced.

Proof: If Φ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$.

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ *be a* $\mathbb{T}$*-gain (connected) graph, then* $\rho(A(\Phi)) = \rho(A(G))$ *if and only if either* Φ *or* $-\Phi$ *is balanced.*

Proof: If Φ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$. Conversely, suppose that $\rho(A(\Phi)) = \rho(A(G))$.

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either Φ or $-\Phi$ is balanced.

Proof: If Φ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$. Conversely, suppose that $\rho(A(\Phi)) = \rho(A(G))$. Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of $A(\Phi)$.

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either Φ or $-\Phi$ is balanced.

Proof: If Φ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$. Conversely, suppose that $\rho(A(\Phi)) = \rho(A(G))$. Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of $A(\Phi)$. Since $A(\Phi)$ is Hermitian, either $\rho(A(\Phi)) = \lambda_1$ or $\rho(A(\Phi)) = -\lambda_n$.

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either Φ or $-\Phi$ is balanced.

Proof: If Φ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$. Conversely, suppose that $\rho(A(\Phi)) = \rho(A(G))$. Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of $A(\Phi)$. Since $A(\Phi)$ is Hermitian, either $\rho(A(\Phi)) = \lambda_1$ or $\rho(A(\Phi)) = -\lambda_n$.

Case 1: Suppose that $\rho(A(\Phi)) = \lambda_1$. Then there is a diagonal unitary matrix $D \in \mathbb{C}^{n\times n}$ such that $A(\Phi) = DA(G)D^{-1}$. Hence $\Phi \sim (G, 1)$. Therefore, Φ is balanced.

Key theorem

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either Φ or $-\Phi$ is balanced.

Proof: If Φ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$. Conversely, suppose that $\rho(A(\Phi)) = \rho(A(G))$. Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of $A(\Phi)$. Since $A(\Phi)$ is Hermitian, either $\rho(A(\Phi)) = \lambda_1$ or $\rho(A(\Phi)) = -\lambda_n$.

Case 1: Suppose that $\rho(A(\Phi)) = \lambda_1$. Then there is a diagonal unitary matrix $D \in \mathbb{C}^{n \times n}$ such that $A(\Phi) = DA(G)D^{-1}$. Hence $\Phi \sim (G, 1)$. Therefore, Φ is balanced.

Case 2: If $\rho(A(\Phi)) = -\lambda_n$, then $\lambda_n = e^{\iota\pi}\rho(A(\Phi))$. We have $A(\Phi) = e^{\iota\pi}DA(G)D^{-1}$, for some diagonal unitary matrix $D \in \mathbb{C}^{n \times n}$. Thus $A(-\Phi) = DA(G)D^{-1}$. Hence, $(-\Phi) \sim (G, 1)$. Thus, $-\Phi$ is balanced.

Converse

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain(connected) graph. Then, $\sigma(A(\Phi)) = \sigma(A(G))$ if and only if Φ is balanced.

Characterization of bipartite graphs

Theorem

Let G be a connected graph. Then, G is bipartite if and only if $\rho(A(\Phi)) = \rho(A(G))$ implies $\sigma(A(\Phi)) = \sigma(A(G))$ for every gain φ.

Characterization of bipartite graphs

Theorem

Let G be a connected graph. Then, G is bipartite if and only if $\rho(A(\Phi)) = \rho(A(G))$ implies $\sigma(A(\Phi)) = \sigma(A(G))$ for every gain φ.

Theorem

Let G be a connected graph. Then

(i) *If G is bipartite, then whenever Φ is balanced implies $-\Phi$ is balanced.*

Characterization of bipartite graphs

Theorem

Let G be a connected graph. Then, G is bipartite if and only if $\rho(A(\Phi)) = \rho(A(G))$ *implies* $\sigma(A(\Phi)) = \sigma(A(G))$ *for every gain* φ.

Theorem

Let G be a connected graph. Then

(i) *If G is bipartite, then whenever* Φ *is balanced implies* $-\Phi$ *is balanced.*

(ii) *If* Φ *is balanced implies* $-\Phi$ *is balanced for some gain* Φ*, then the graph is bipartite.*

Invariance of gain spectrum and gain spectral radius

Theorem

Let $\Phi = (G, \varphi)$ *be a* $\mathbb{T}$*-gain graph. Then G is a tree if and only if* $\sigma(A(G)) = \sigma(A(\Phi))$ *for all* φ.

Invariance of gain spectrum and gain spectral radius

Theorem

Let $\Phi = (G, \varphi)$ *be a* $\mathbb{T}$*-gain graph. Then G is a tree if and only if* $\sigma(A(G)) = \sigma(A(\Phi))$ *for all* φ.

Theorem

Let $\Phi = (G, \varphi)$ *be a* $\mathbb{T}$*-gain graph. Then G is a tree* $\Leftrightarrow$ $\rho(A(G)) = \rho(A(\Phi))$ *for all* φ.

Invariance of gain spectrum and gain spectral radius

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. Then G is a tree if and only if $\sigma(A(G)) = \sigma(A(\Phi))$ for all φ.

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. Then G is a tree $\Leftrightarrow$ $\rho(A(G)) = \rho(A(\Phi))$ for all φ.

Theorem

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. TFAE,

1. *G is tree,*
2. *$\sigma(A(G)) = \sigma(A(\Phi))$ for all φ,*
3. *$\rho(A(G)) = \rho(A(\Phi))$ for all φ.*

References I

- R. B. Bapat, *Graphs and matrices*, Universitext, Springer, London; Hindustan Book Agency, New Delhi, 2010. MR 2797201

- R. B. Bapat and Sukanta Pati, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc. **1** (2004), no. 2, 129–132. MR 2250987

- Andries E. Brouwer and Willem H. Haemers, *Spectra of graphs*, Universitext, Springer, New York, 2012. MR 2882891

- Gilles Caporossi, Dragos Cvetkovic, Ivan Gutman, and Pierre Hansen, *Variable neighborhood search for extremal graphs. 2. finding graphs with extremal energy*, Journal of Chemical Information and Computer Sciences **39** (1999), no. 6, 984–996.

- M. Cavers, S. M. Cioabă, S. Fallat, D. A. Gregory, W. H. Haemers, S. J. Kirkland, J. J. McDonald, and M. Tsatsomeros, *Skew-adjacency matrices of graphs*, Linear Algebra Appl. **436** (2012), no. 12, 4512–4529. MR 2917427

References II

- Krystal Guo and Bojan Mohar, *Hermitian adjacency matrix of digraphs and mixed graphs*, J. Graph Theory **85** (2017), no. 1, 217–248. MR 3634484

- Karl E. Gustafson and Duggirala K. M. Rao, *Numerical range*, Universitext, Springer-Verlag, New York, 1997, The field of values of linear operators and matrices. MR 1417493

- I. Gutman, *The energy of a graph*, Ber. Math.-Statist. Sekt. Forshungszentrum Graz **103** (1978), no. 2, 1–22.

- Ivan Gutman, *The energy of a graph: Old and new results*, Algebraic Combinatorics and Applications (Berlin, Heidelberg) (Anton Betten, Axel Kohnert, Reinhard Laue, and Alfred Wassermann, eds.), Springer Berlin Heidelberg, 2001, pp. 196–211.

- Frank Harary, *The determinant of the adjacency matrix of a graph*, SIAM Review **4** (Jul., 1962), no. 1, 202–210.

References III

- A. J. Hoffman, *On eigenvalues and colorings of graphs*, Graph Theory and its Applications, Academic, New York and London (1970), 79–91.
- Roger A. Horn and Charles R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013. MR 2978290
- O.E. Polansky. I. Gutman, *Mathematical concepts in organic chemistry*, Springer Berlin Heidelberg, 1986.
- Jianxi Liu and Xueliang Li, *Hermitian-adjacency matrices and Hermitian energies of mixed graphs*, Linear Algebra Appl. **466** (2015), 182–207. MR 3278246
- R Mehatari, M Rajesh Kannan, and A Samanta, *On the adjacency matrix of complex unit gain graphs*, (Submitted, arXiv:1812.03747) (2018).
- Juan Rada and Antonio Tineo, *Upper and lower bounds for the energy of bipartite graphs*, J. Math. Anal. Appl. **289** (2004), no. 2, 446–455. MR 2026917

References IV

Nathan Reff, *Spectral properties of complex unit gain graphs*, Linear Algebra Appl. **436** (2012), no. 9, 3165–3176. MR 2900705

H. S. Wilf, *The eigenvalues of a graph and its chromatic number*, Journal of the London Mathematical Society **s1-42** (1967), no. 1, 330–332.

Thomas Zaslavsky, *Biased graphs. I. Bias, balance, and gains*, J. Combin. Theory Ser. B **47** (1989), no. 1, 32–52. MR 1007712