

Problem Sheet – 3

Multivariable calculus

- Try to solve all the problems.
- No need to submit the answers to the instructors.

1. Find the following limits:

- (a) $\lim_{(x,y) \rightarrow (0, \ln 2)} e^{x-y}.$
- (b) $\lim_{(x,y) \rightarrow (1,1)} \ln |1 + x^2 y^2|.$
- (c) $\lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x}.$
- (d) $\lim_{(x,y) \rightarrow (1/27, \sqrt{3})} \cos \sqrt[3]{xy}.$

2. Find the following limits ($\mathbf{p} := (x, y, z)$):

- (a) $\lim_{\mathbf{p} \rightarrow (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right).$
- (b) $\lim_{\mathbf{p} \rightarrow (1,-1,-1)} \frac{2xy + yz}{x^2 + z^2}.$
- (c) $\lim_{\mathbf{p} \rightarrow (\pi, \pi, 0)} (\sin^2 x + \cos^2 y + \sec^2 z).$
- (d) $\lim_{\mathbf{p} \rightarrow (-1/4, \pi/2, 2)} \tan^{-1}(xyz).$
- (e) $\lim_{\mathbf{p} \rightarrow (\pi, 0, 3)} z e^{-2y} \cos 2x.$

3. In the following exercise, for the given ϵ , find a $\delta > 0$ such that

$$\sqrt{x^2 + y^2} < \delta \implies |f(x, y) - f(0, 0)| < \epsilon.$$

- (a) $f(x, y) = x^2 + y^2, \quad \epsilon = 0.01.$
- (b) $f(x, y) = \frac{y}{x^2 + 1}, \quad \epsilon = 0.05.$
- (c) $f(x, y) = \frac{x + y}{x^2 + 1}, \quad \epsilon = 0.01.$
4. Find all the second-order partial derivatives of the functions.
- (a) $g(x, y) = x^2 y + \cos y + y \sin x.$
- (b) $h(x, y) = x e^y + y + 1.$
- (c) $r(x, y) = \ln(x + y).$
- (d) $s(x, y) = \tan^{-1} \left(\frac{y}{x} \right).$
5. Use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.
- (a) $f(x, y) = 1 - x + y - 3xy, \quad \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ at } (1, 2).$
- (b) $f(x, y) = \sqrt{2x + 3y - 1}, \quad \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ at } (2, 3).$
6. (a) Find $\frac{\partial w}{\partial r}$ when $r = 1, s = -1$, if

$$w = (x + y + z)^2, \quad x = r - s, \quad y = \cos(r + s), \quad z = \sin(r + s).$$

- (b) Find $\frac{\partial w}{\partial v}$ when $u = -1$, $v = 2$, if

$$w = xy + \ln z, \quad x = \frac{u^2}{v}, \quad y = u + v, \quad z = \cos u.$$

7. Find the derivative of the function at P_0 in the direction of the vector $\mathbf{u}$.
- (a) $f(x, y) = x^2 - 3y^2$, $P_0 = (5, 5)$, $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$.
 - (b) $f(x, y) = 2x^2 + y^2$, $P_0 = (-1, 1)$, $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.
 - (c) $g(x, y) = \frac{x - y}{xy + 2}$, $P_0 = (1, -1)$, $\mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$.
 - (d) $h(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + \sqrt{3}\sin^{-1}\left(\frac{xy}{2}\right)$, $P_0 = (1, 1)$, $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$.
 - (e) $f(x, y, z) = xy + yz + zx$, $P_0 = (1, -1, 2)$, $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.
8. Find the local maxima, local minima, and saddle points of the functions.
- (a) $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$.
 - (b) $f(x, y) = x^2 - 4xy + y^2 - 6x + 2$.
 - (c) $f(x, y) = 2x^2 + 3xy + 4y^2 - 2x + 2y$.
 - (d) $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$.
 - (e) $f(x, y) = 1 - \sqrt{x^2 + y^2}$.
9. Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines

$$x = 0, \quad y = 0, \quad y = 9 - x.$$

10. Find the absolute maxima and minima of the functions on the given domains.
- (a) $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant.
 - (b) $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 3$.
 - (c) $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$.
 - (d) $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \leq x \leq 1$, $0 \leq y \leq 1$.
11. (a) Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.
- (b) Find the points on the curve $xy^2 = 54$ nearest the origin.
 - (c) Find the points on the curve $x^2y = 2$ nearest the origin.
 - (d) Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.
 - (e) Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(-1, -1, 1)$.
 - (f) Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.
 - (g) Find the point on the surface $z = xy + 1$ nearest the origin.