Problem Sheet – 1
Riemann Integration

- Try to solve all the problems by 11-09-2026.
- No need to submit the answers to the instructors.

Riemann Sums and the Definition of the Integral

Problem 1. For $f(x) = x$ on $[0, 1]$, divide the interval into n equal subintervals and use the right endpoints to form a Riemann sum. Evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$

and hence find

$$\int_0^1 x\,dx.$$

Problem 2. Express the integral

$$\int_0^2 x^2\,dx$$

as the limit of a Riemann sum using right endpoints. Evaluate the resulting limit.

Problem 3. Express

$$\int_1^3 \sqrt{x}\,dx$$

as a limit of Riemann sums using left endpoints.

Problem 4. Use a Riemann sum with n equal subintervals and right endpoints to evaluate

$$\int_0^1 (3x^2 + 2x + 1)\,dx.$$

Problem 5. Evaluate the following limit by interpreting it as a Riemann sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1 + \frac{i}{n}\right)^3.$$

Riemann Integrability

Problem 6. Determine whether each of the following functions is Riemann integrable on the indicated interval. Justify your answer.

(a) $f(x) = x^2$ on $[0, 1]$;

(b) $f(x) = |x - 1|$ on $[0, 2]$;

(c)

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad [0, 1];$$

(d)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases} \qquad [0,1].$$

Problem 7. Let

$$f(x) = \begin{cases} 1, & x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether f is Riemann integrable on $[0,1]$.

Problem 8. Let f be a bounded function on $[a,b]$ such that $f(x) = g(x)$ except at finitely many points, where g is Riemann integrable. Explain why f is Riemann integrable and show that

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Problem 9. Show that if f is Riemann integrable on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$.

Properties of the Riemann Integral

Problem 10. Assume that f and g are Riemann integrable on $[a,b]$.

(a) If c is a constant, show that cf is Riemann integrable and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(c) Prove that $f+g$ is Riemann integrable on $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(d) Suppose that

$$f(x) \le g(x) \qquad \text{for every } x \in [a,b].$$

Prove that

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

(e) If

$$m \le f(x) \le M \qquad \text{for all } x \in [a,b],$$

show that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

(f) Show that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Problem 11. Suppose that f is integrable on $[-a,a]$. Prove the following:

(a) If f is an odd function, then

$$\int_{-a}^a f(x)\,dx = 0.$$

(b) If f is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Average Value of a Function

If f is integrable on $[a, b]$, then its **average value** on $[a, b]$ is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problem 12. Find the average value of each of the following functions on the given interval.

(a)
$$f(x) = x^2, \qquad x \in [0, 3].$$

(b)
$$f(x) = \sin x, \qquad x \in [0, \pi].$$

Problem 13. Suppose
$$|f(x)| \le 3 \qquad \text{for } 0 \le x \le 2.$$

Find an upper bound for
$$\left|\int_0^2 f(x)\,dx\right|.$$

Areas Between Curves

Problem 14. Find the total area of the region between the graph of f and the x-axis.

(a) $f(x) = x^2 - 4x + 3, \qquad 0 \le x \le 3.$

(b) $f(x) = 1 - \dfrac{x^2}{4}, \qquad -2 \le x \le 3.$

(c) $f(x) = 5 - 5x^{2/3}, \qquad -1 \le x \le 8.$

Problem 15. Find the area of the region enclosed by the given curves and lines.

(a) $y = x, \qquad y = \dfrac{1}{x^2}, \qquad x = 2.$

(b) $y = x, \qquad y = \dfrac{1}{\sqrt{x}}, \qquad x = 2.$

(c) $\sqrt{x} + \sqrt{y} = 1, \qquad x = 0, \qquad y = 0.$

Problem 16. Find the area of the region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

Problem 17. Find the total area of the region between the curves

$$y = \sin x \qquad \text{and} \qquad y = \cos x,$$

for $0 \le x \le \dfrac{3\pi}{2}$.

Problem 18. Find the area between the curve

$$y = \frac{2\ln x}{x}$$

and the x-axis from $x = 1$ to $x = e$.

Volumes of Solids with Known Cross-Sections

Problem 19. Find the volume of each solid.

(a) The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the x-axis on $0 \leq x \leq 4$ are squares with diagonals running from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(b) The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are squares whose bases run from the semicircle

$$y = -\sqrt{1 - x^2}$$

to the semicircle

$$y = \sqrt{1 - x^2}.$$

(c) The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x-axis are:

(i) circular disks with diameters running from $y = \tan x$ to $y = \sec x$;

(ii) squares whose bases run from $y = \tan x$ to $y = \sec x$.

Volumes of Revolution

Problem 20. Find the volumes of the solids generated by revolving the regions bounded by the given lines and curves about the x-axis.

(a) $y = x^2, \qquad y = 0, \qquad x = 2.$

(b) $y = \sqrt{\cos x}, \qquad 0 \leq x \leq \frac{\pi}{2}, \qquad y = 0, \qquad x = 0.$

(c) The region between the curve

$$y = \sqrt{x \cot x}$$

and the x-axis from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{2}$.

Problem 21. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the y-axis.

(a) The region enclosed by

$$x = \sqrt{2y}, \qquad x = 0, \qquad y = -1, \qquad y = 1.$$

(b) The region enclosed by

$$x = \frac{2}{\sqrt{y+1}}, \qquad x = 0, \qquad y = 0, \qquad y = 3.$$

(c) The region enclosed by

$$x = \sqrt{\frac{2y}{y^2+1}}, \qquad x = 0, \qquad y = 1.$$

Happy Weekend!!!
