

Lecture 13

Constrained Extrema: Lagrange Multipliers

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భారతీయ పాఠశాల విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Constrained Optimisation

The Method of Lagrange Multipliers

Worked Examples

Practice and Summary

Extrema Subject to a Constraint

- So far we have looked for extrema of f over an open region. Often the variables are **tied together** by a condition: find the extreme values of $f(x, y)$ (or $f(x, y, z)$) only over points satisfying a **constraint** $g = 0$.

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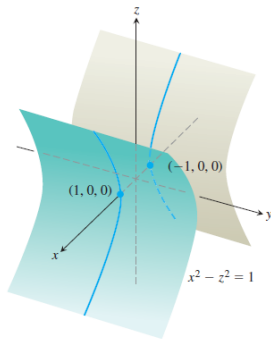
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Substitution Can Mislead

Problem

Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ closest to the origin.



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- **What went wrong?** h is defined on the whole xy -plane, but only points with $|x| \geq 1$ come from the cylinder. We optimised over the wrong domain.

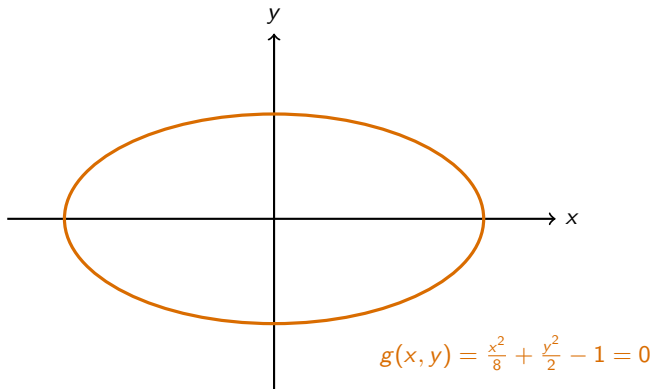
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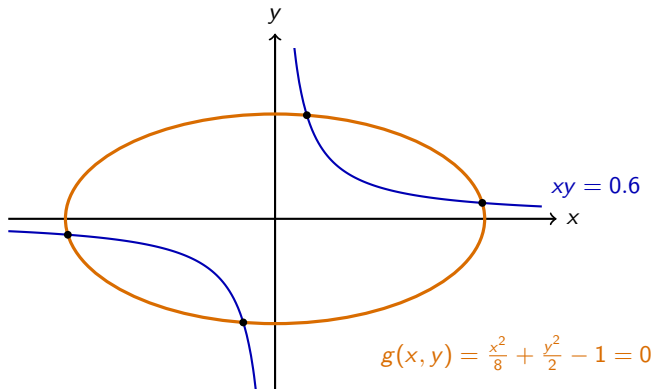
Moral: substitution works, but it depends on a lucky choice of variable, and it can quietly change the domain. We want a method that treats the constraint *as it is*.

Level Curves Touching the Constraint: Step by Step



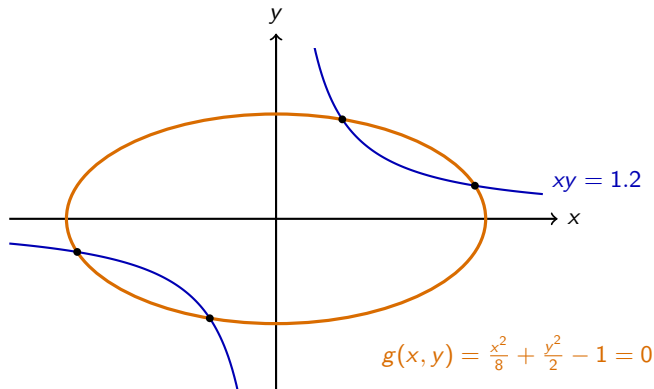
Goal: the largest and smallest values of $f(x, y) = xy$ at points of the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ (orange).

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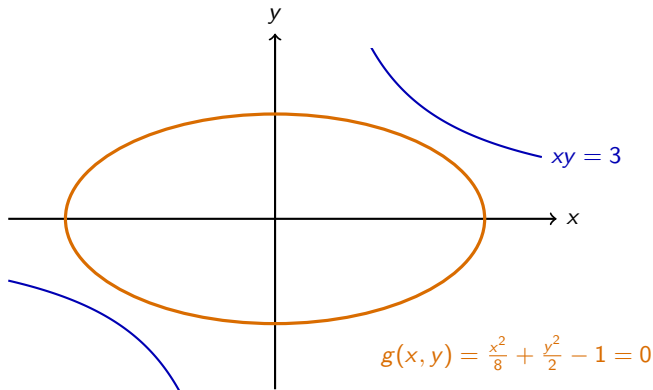
The level curve $xy = 0.6$ **crosses** the ellipse. Moving along the ellipse past a crossing point takes us to larger values of xy , so 0.6 is not the maximum.

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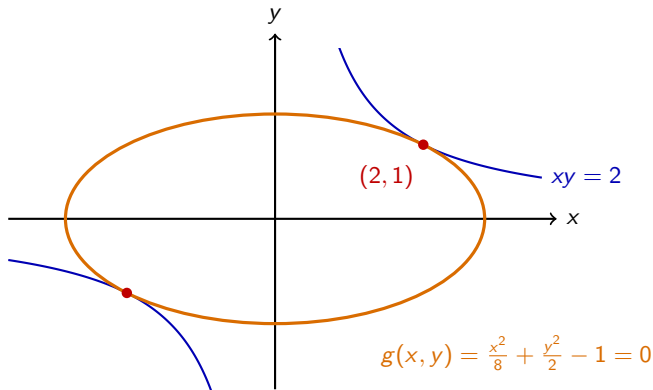
$xy = 1.2$ still crosses. As long as a level curve cuts across the constraint, f can still be increased along it.

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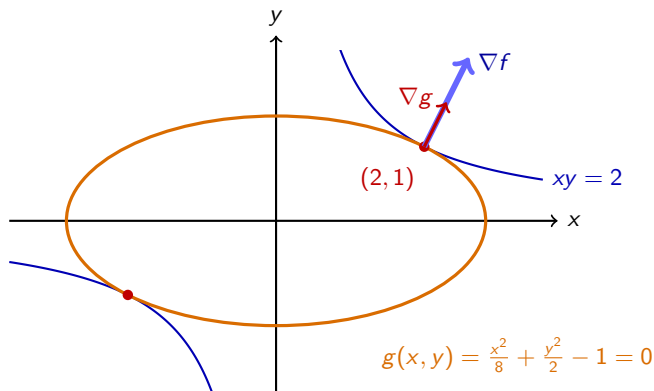
$xy = 3$ misses the ellipse entirely: no point of the constraint has $f = 3$.
The maximum is somewhere in between.

Level Curves Touching the Constraint: Step by Step



$xy = 2$ just **touches** the ellipse, at $(2, 1)$ and $(-2, -1)$: the largest value of f on the constraint is 2. At the touching point the two curves are *tangent*.

Level Curves Touching the Constraint: Step by Step



Tangent curves have parallel normals, and gradients are normal to level curves: at $(2, 1)$, $\nabla f = \mathbf{i} + 2\mathbf{j}$ and $\nabla g = \frac{1}{2}\mathbf{i} + \mathbf{j}$, so $\nabla f = 2\nabla g$. **This is the whole idea.**

Why the Gradients Must Be Parallel

Theorem (Orthogonal Gradient Theorem)

Suppose $f(x, y)$ is differentiable in a region whose interior contains a smooth curve $C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. If P_0 is a point on C where f has a local maximum or minimum relative to its values on C , then ∇f is orthogonal to C at P_0 .

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- Take C to be the constraint curve $g(x, y) = 0$. By Lecture 10, ∇g is also orthogonal to C .
- Two vectors orthogonal to the same tangent line are parallel:
 $\nabla f = \lambda \nabla g$ (if $\nabla g \neq \mathbf{0}$).

The Method of Lagrange Multipliers

Theorem (Lagrange Multipliers, one constraint)

Suppose f and g are differentiable and $\nabla g \neq \mathbf{0}$ at points where $g = 0$. To find the local maximum and minimum values of f subject to the constraint $g = 0$ (if these exist), find the values of the variables and of λ that simultaneously satisfy

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- **Three variables:** $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, $g(x, y, z) = 0$ — four equations in x, y, z, λ . (Here $g = 0$ is a surface, and the same argument applies to every curve on it through P_0 .)

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- The number λ is the **Lagrange multiplier**. We usually eliminate it; we rarely need its value.
- **Recipe:** solve the system, then *evaluate f at every solution*: the largest value is the maximum, the smallest the minimum (when these exist, e.g. on a closed bounded curve such as an ellipse).

Worked Example: xy on an Ellipse

Problem

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- So $\lambda = \pm 2$ and $x = \pm 2y$. Then $\frac{4y^2}{8} + \frac{y^2}{2} = 1$ gives $y^2 = 1$:

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- $f = xy = 2$ at $(2, 1)$, $(-2, -1)$ and $f = -2$ at $(2, -1)$, $(-2, 1)$.

Greatest value 2, smallest -2 — matching the tangent level curves in the picture.

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- Substituting into $x^2 + y^2 = 1$: $\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$, so $\lambda^2 = \frac{25}{4}$, $\lambda = \pm \frac{5}{2}$.

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Solution

- $g = x^2 + y^2 - 1$, so $\nabla f = \lambda \nabla g$ reads $3 = 2\lambda x$, $4 = 2\lambda y$. So $\lambda \neq 0$ and $x = \frac{3}{2\lambda}$, $y = \frac{2}{\lambda}$.
- Substituting into $x^2 + y^2 = 1$: $\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$, so $\lambda^2 = \frac{25}{4}$, $\lambda = \pm \frac{5}{2}$.
- $\lambda = \frac{5}{2}$: the point $(\frac{3}{5}, \frac{4}{5})$, where $f = 5$. $\lambda = -\frac{5}{2}$: the point $(-\frac{3}{5}, -\frac{4}{5})$, where $f = -5$.

Worked Example: A Linear Function on a Circle

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Geometric check: the level curves $3x + 4y = c$ are parallel lines, and the extreme ones touch the circle where the radius points along $\nabla f = 3\mathbf{i} + 4\mathbf{j}$ — exactly at $\pm \frac{1}{5}(3, 4)$.

Worked Example: Three Variables

Problem

Use Lagrange multipliers to find the points on $x^2 - z^2 - 1 = 0$ closest to the origin.

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Same answer as the careful substitution — but with no choice of variable to get right, and no hidden change of domain.

- Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 8$.

Practice

- Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 8$.
- Find the maximum and minimum values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Practice

- Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 8$.
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- Find the point on the plane $x + 2y + 2z = 9$ closest to the origin.

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- Find the maximum and minimum values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.
- Find the point on the plane $x + 2y + 2z = 9$ closest to the origin.
- Find the largest area of a rectangle with sides parallel to the axes inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. (*Hint: maximise $4xy$.*)

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- The same equations work in three variables, with a constraint surface $g(x, y, z) = 0$.
- The key ingredients all came from earlier lectures: the chain rule, and the gradient being normal to level curves.