Lecture 12
The Second Derivative Test and Saddle Points

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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Outline

The Second Derivative Test

Worked Examples

Practice and Summary

Where We Are

- **Last lecture:** at an interior local extremum where the partials exist, $f_x = f_y = 0$. Solving these gives the *candidates* — the critical points.

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- The right combination turns out to be a single number, the discriminant.

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Theorem (Second Derivative Test for Local Extreme Values)

Suppose $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centred at (a, b), and $f_x(a, b) = f_y(a, b) = 0$. Then

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The quantity

$$D = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

is the **discriminant** (or **Hessian**) of f.

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is the **discriminant** (or **Hessian**) of f. Order of work: first the sign of D; only if $D > 0$ look at the sign of f_{xx}.

Why the Discriminant? (Intuition)

- Near a critical point the first-order terms vanish, so the *quadratic* part of f takes over. Writing $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$ at (a, b), for small h, k:

$$f(a+h, b+k) - f(a, b) \;\approx\; \tfrac{1}{2}\big(Ah^2 + 2Bhk + Ck^2\big).$$

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- If $A \neq 0$, complete the square:

$$Q := Ah^2 + 2Bhk + Ck^2 \;=\; A\left[\left(h + \frac{B}{A}k\right)^2 + \frac{AC - B^2}{A^2}\,k^2\right].$$

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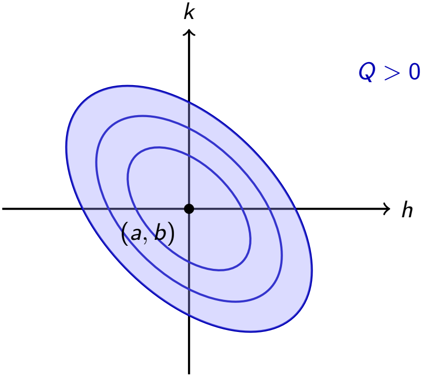
- If $AC - B^2 = 0$: the quadratic part vanishes along a whole line, and the higher-order terms (which we ignored) decide.

Reading the Quadratic Part: Step by Step



$D > 0$, $f_{xx} > 0$: the level curves of $Q = Ah^2 + 2Bhk + Ck^2$ are **ellipses** and $Q > 0$ all round. Moving off in *any* direction increases f: a local minimum.

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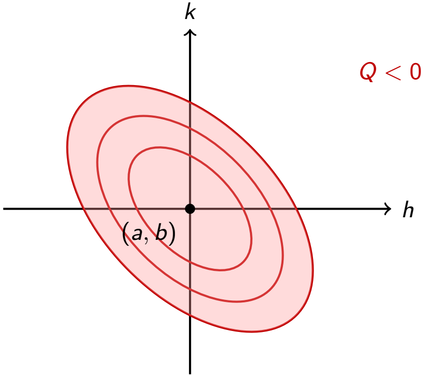

$D > 0$, $f_{xx} < 0$: the same ellipses, but now $Q < 0$ all round. Every direction decreases f: a local maximum.

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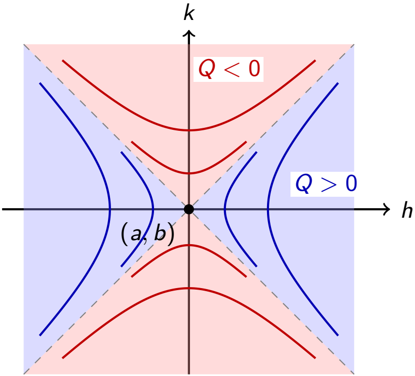

$D < 0$: the level curves are **hyperbolas**. $Q > 0$ in two opposite sectors and $Q < 0$ in the other two, so f goes up one way and down another: a saddle point.

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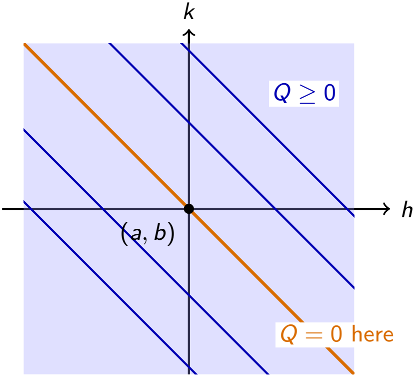

$D = 0$: Q is a perfect square, zero along a whole line through (a, b). Along that line the quadratic part says *nothing*; higher-order terms decide, so the test is inconclusive.

Worked Example: A Local Maximum

Problem

Find the local extreme values of $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.

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- f is a polynomial, defined and differentiable everywhere. Critical points:

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0 \quad \Longrightarrow \quad x = y = -2.$$

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- Second partials: $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 1$, so

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- $D > 0$ and $f_{xx} < 0$: a **local maximum** at $(-2, -2)$, with value $f(-2, -2) = 8$.

Worked Example: Finishing Last Lecture's Problem

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Classify the critical points $(0, 0)$ and $(1, 1)$ of $f(x, y) = x^3 + y^3 - 3xy$.

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- From $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$:

$$f_{xx} = 6x, \qquad f_{yy} = 6y, \qquad f_{xy} = -3, \qquad D = 36xy - 9.$$

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- At $(0,0)$: $D = -9 < 0$, so $(0,0)$ is a **saddle point**.
- At $(1,1)$: $D = 36 - 9 = 27 > 0$ and $f_{xx} = 6 > 0$, so f has a **local minimum** there, $f(1,1) = -1$.

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Note that D is a *function* of (x, y): compute it once in general, then evaluate at each critical point.

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Solution

- $f_x = -6x + 6y = 0$ gives $y = x$. Then $f_y = 6y - 6y^2 + 6x = 12x - 6x^2 = 6x(2 - x) = 0$, so $x = 0$ or $x = 2$. Critical points: $(0, 0)$ and $(2, 2)$.

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- $f_{xx} = -6$, $f_{yy} = 6 - 12y$, $f_{xy} = 6$, so $D = -6(6 - 12y) - 36 = 72(y - 1)$.

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- $f_{xx} = -6$, $f_{yy} = 6 - 12y$, $f_{xy} = 6$, so $D = -6(6 - 12y) - 36 = 72(y - 1)$.
- At $(0, 0)$: $D = -72 < 0$: a **saddle point**.

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- $f_{xx} = -6$, $f_{yy} = 6 - 12y$, $f_{xy} = 6$, so $D = -6(6 - 12y) - 36 = 72(y - 1)$.
- At $(0, 0)$: $D = -72 < 0$: a **saddle point**.
- At $(2, 2)$: $D = 72 > 0$ and $f_{xx} = -6 < 0$: a **local maximum**, $f(2, 2) = 12 - 16 - 12 + 24 = 8$.

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- But $f_1 \geq 0 = f_1(0, 0)$: a **minimum**; $f_2 \leq 0$: a **maximum**; $f_3(x, 0) = x^4 > 0$ and $f_3(0, y) = -y^4 < 0$: a **saddle**.

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Moral: when $D = 0$, all three outcomes are possible, and we must go back to the definitions (or look at f along suitable curves).

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Moral: when $D = 0$, all three outcomes are possible, and we must go back to the definitions (or look at f along suitable curves).

Also: the test needs $f_x = f_y = 0$ *first*. Computing D at a point that is not critical tells you nothing about extrema.

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- Classify the critical points of $g(x, y) = x^3 + y^3 - 6xy$. (You found them in the Lecture 11 Practice.)
- Find and classify the critical points of $f(x, y) = x^2 - y^2 - 2x + 4y + 6$.

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- Find and classify the critical points of $f(x,y) = x^2 - y^2 - 2x + 4y + 6$.
- Show that the second derivative test is inconclusive for $f(x,y) = x^2y^2$ at the origin, and then decide the nature of the origin directly.

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- $D > 0,\ f_{xx} < 0$: **local max**. $D > 0,\ f_{xx} > 0$: **local min**.
- $D < 0$: **saddle point**. $D = 0$: **inconclusive —** go back to the definitions.
- *Why:* near a critical point f behaves like its quadratic part $\frac{1}{2}(Ah^2 + 2Bhk + Ck^2)$, whose level curves are ellipses ($D > 0$), hyperbolas ($D < 0$) or parallel lines ($D = 0$).

Summary

- At a critical point, compute the **discriminant** $D = f_{xx} f_{yy} - f_{xy}^2$.
- $D > 0,\ f_{xx} < 0$: **local max**. $D > 0,\ f_{xx} > 0$: **local min**.
- $D < 0$: **saddle point**. $D = 0$: **inconclusive —** go back to the definitions.
- *Why:* near a critical point f behaves like its quadratic part $\frac{1}{2}(Ah^2 + 2Bhk + Ck^2)$, whose level curves are ellipses ($D > 0$), hyperbolas ($D < 0$) or parallel lines ($D = 0$).
- **Next lecture:** extrema subject to a *constraint* — Lagrange multipliers.