Lecture 11
Local Extreme Values of Functions of Two Variables

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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Outline

Optimisation, One Variable and Two

- **One variable.** To find the local extreme values of $f(x)$ we solved $f'(x) = 0$ to get the *candidates*, then used the sign of f'' to decide which were maxima and which minima.

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- **Two variables.** The same two-stage plan works. Today we build the first stage: a necessary condition that produces the candidate points.
- **Next lecture:** the second derivative test, which sorts the candidates out.

Local Maximum and Local Minimum

Definition

Let $f(x, y)$ be defined on a region R containing the point (a, b).

- $f(a, b)$ is a **local maximum value** of f if $f(a, b) \geq f(x, y)$ for all (x, y) in some open disk centred at (a, b).

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- The comparison is with *nearby* points only — that is what "local" means, and why an open disk appears rather than the whole region.
- If the inequality holds for *all* (x, y) in R, the value is an **absolute** (or global) maximum or minimum on R.
- Geometrically: a local maximum is a hilltop on the surface $z = f(x, y)$; a local minimum is the bottom of a hollow.

Where Can a Local Extremum Occur?

Theorem (First Derivative Test for Local Extreme Values)

If $f(x, y)$ has a local maximum or a local minimum at an ***interior point*** *(a, b) of its domain, and if f_x and f_y exist at (a, b), then*

$$f_x(a, b) = 0 \qquad \textit{and} \qquad f_y(a, b) = 0.$$

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Suppose f has a local maximum at (a, b). Freeze $y = b$: the one-variable function $g(x) = f(x, b)$ has an ordinary local maximum at the interior point $x = a$, and $g'(a) = f_x(a, b)$ exists.

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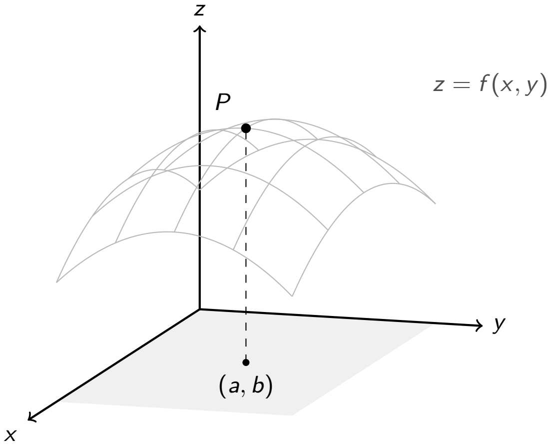
In gradient language: $\nabla f(a, b) = \mathbf{0}$, so $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0$ in *every* direction $\mathbf{u}$ — the surface is "level" at (a, b), whichever way we set off.

Horizontal Does Not Mean Extreme: Step by Step



A hilltop: $P = (a, b, f(a, b))$ sits at a local maximum of the surface $z = f(x, y)$.

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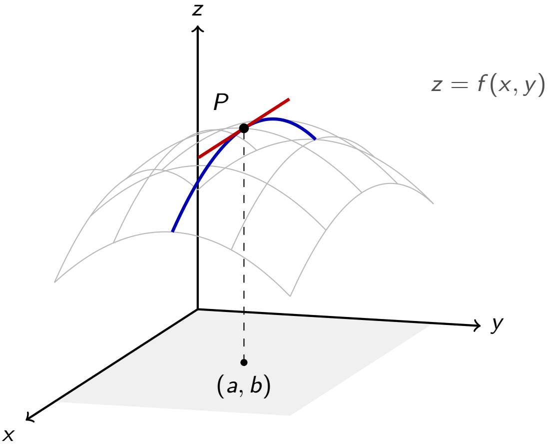

Freeze $y = b$. The trace in the x-direction (blue) has a peak at P, so its tangent (red) is horizontal: $f_x(a, b) = 0$.

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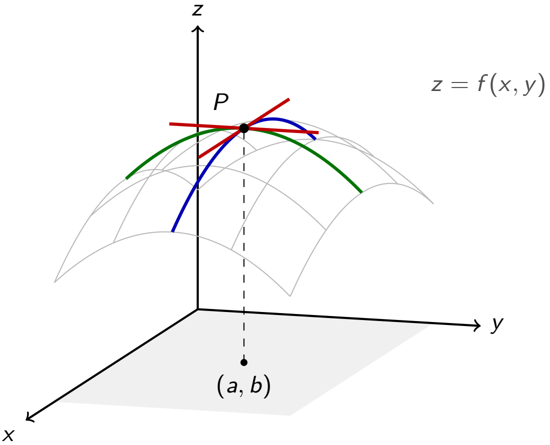

Freeze $x = a$. The trace in the y-direction (green) also peaks at P: $f_y(a, b) = 0$.

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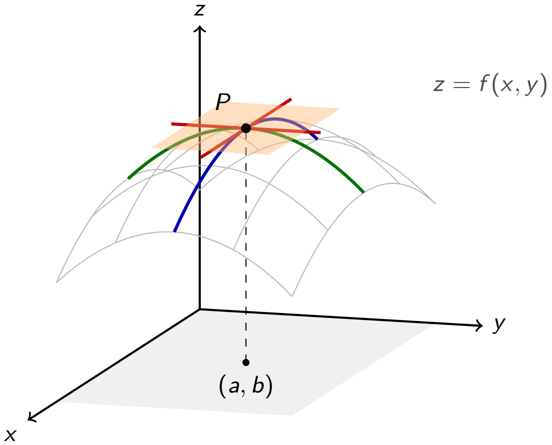

Both tangent lines are horizontal, so the surface is level at P in every direction — and here P really is a local maximum.

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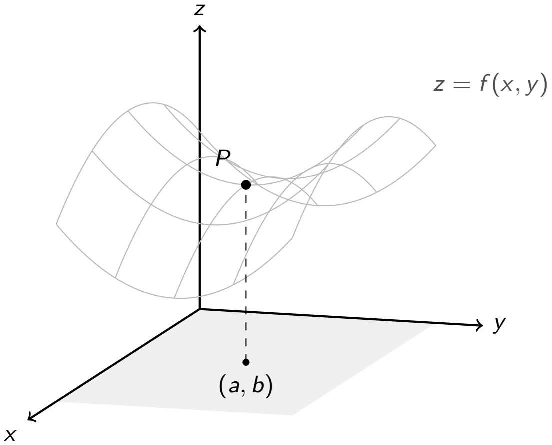

Now a different surface, again with a point P above (a, b).

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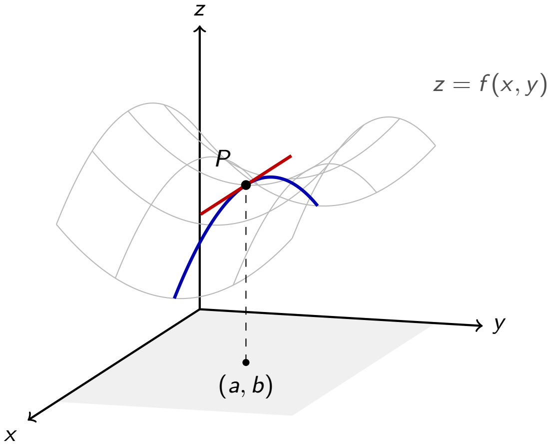

The x-trace (blue) has a **peak** at P: horizontal tangent, $f_x(a, b) = 0$.

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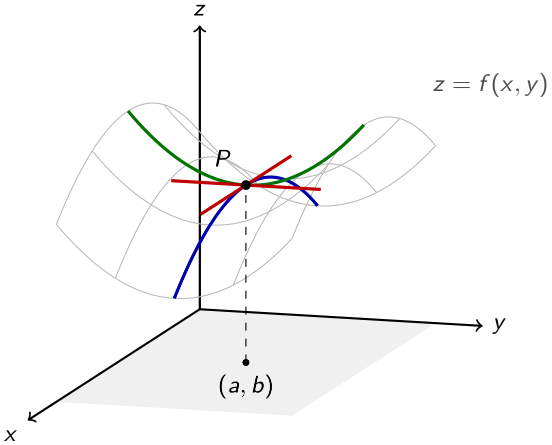

But the y-trace (green) has a **valley** at P: again a horizontal tangent, $f_y(a, b) = 0$.

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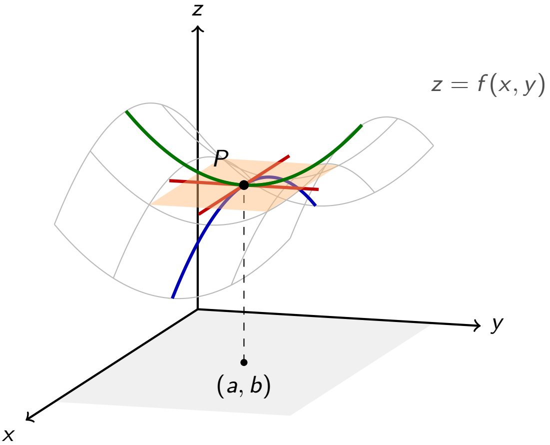

So $f_x = f_y = 0$ in *both* pictures, yet the second surface rises above the level plane one way and dips below it the other: a saddle, not an extremum. The condition finds *candidates*; it does not classify them.

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- **The second clause matters.** $f(x, y) = \sqrt{x^2 + y^2}$ has a local (in fact absolute) minimum at the origin — a cone's tip — where *neither* partial exists. The First Derivative Test says nothing there, yet the point is still a candidate.

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- Consequently: a local extreme value of f can occur **only** at a critical point or at a boundary point of the domain. Those are the only places to look.

Saddle Points

Definition

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if every open disk centred at (a, b) contains domain points (x_1, y_1) and (x_2, y_2) with

$$f(x_1, y_1) > f(a, b) \qquad \text{and} \qquad f(x_2, y_2) < f(a, b).$$

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The corresponding point $\big(a, b, f(a, b)\big)$ on the surface is a **saddle point of the surface**.

- In words: however closely we look, there are nearby points where f is bigger *and* nearby points where f is smaller. So (a, b) is neither a local maximum nor a local minimum.

Worked Example: A Minimum by Completing the Square

Problem

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- The domain is the whole plane (no boundary points) and f is a polynomial (partials exist everywhere). So the critical points solve

$$f_x = 2x = 0, \qquad f_y = 2y - 4 = 0,$$

giving the single critical point $(0, 2)$, with $f(0, 2) = 5$.

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- To classify it without the second derivative test, complete the square:

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- So $f(0,2) = 5$ is a local minimum — in fact an absolute minimum on the whole plane.

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- Along the y-axis: $f(0, y) = y^2 > 0$ for every $y \neq 0$.
- Every disk about the origin contains points of both kinds, so by the definition $(0, 0)$ is a **saddle point —** and f has *no* local extreme values at all.

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- Every disk about the origin contains points of both kinds, so by the definition $(0, 0)$ is a **saddle point —** and f has *no* local extreme values at all.

This is exactly the second surface in the step-by-step picture: a maximum along the x-direction, a minimum along the y-direction.

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- Hence the critical points are $(0,0)$ and $(1,1)$, with $f(0,0) = 0$ and $f(1,1) = -1$.

Neither completing the square nor a quick inspection settles what kind of points these are. That is precisely what the second derivative test is for — next lecture.

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- Show that $f(x, y) = xy$ has a saddle point at the origin, using the definition directly.
- Find all critical points of $f(x, y) = x^3 + y^3 - 6xy$.
- Explain why $f(x, y) = |x| + |y|$ has a local minimum at the origin even though neither partial derivative exists there. Which clause of the definition of a critical point applies?

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- The converse fails: at a **saddle point** both partials vanish, yet every disk around it contains larger and smaller values.
- So vanishing partials produce *candidates*, not answers — the **second derivative test** (next lecture) does the classifying.