Lecture 10
Directional Derivatives and the Gradient

M. Rajesh Kannan and D. Sukumar

Department of Mathematics,
Indian Institute of Technology Hyderabad,
email: rajeshkannan@math.iith.ac.in, suku@math.iith.ac.in

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Outline

Rates of Change in Any Direction

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- The set-up. Fix a **unit** vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and walk along the line through $P(x_0, y_0)$ in that direction:

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  Because $\mathbf{u}$ is a unit vector, the parameter s measures *arc length* — actual distance travelled — which is what makes the resulting rate comparable across directions.
- Restricted to this line, f becomes a function of the single variable s, and we can differentiate it.

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$$\left(\frac{df}{ds}\right)_{\mathbf{u},P} = \lim_{s\to 0} \frac{f(x_0 + su_1,\ \ y_0 + su_2) - f(x_0, y_0)}{s},$$

provided the limit exists. It is also written $(D_{\mathbf{u}}f)_P$.

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- **The partials are special cases.** Taking $\mathbf{u} = \mathbf{i} = (1, 0)$ gives exactly $f_x(x_0, y_0)$; taking $\mathbf{u} = \mathbf{j} = (0, 1)$ gives $f_y(x_0, y_0)$.

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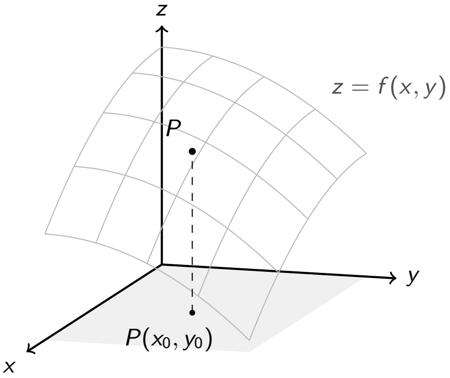
- Geometrically: slice the surface by the vertical plane containing that line, and take the slope of the resulting trace at P — exactly what we did for f_x, but with the slicing plane now free to point anywhere.

Slicing in Any Direction: Step by Step



The surface $z = f(x, y)$, with P sitting above the point (x_0, y_0) of the domain.

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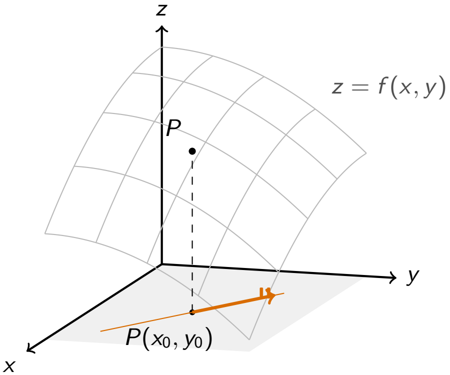
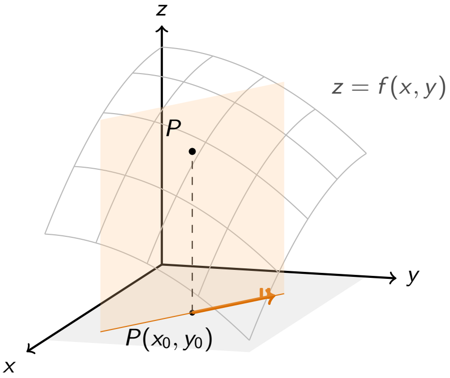


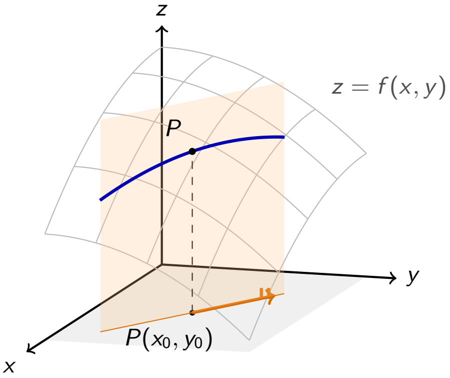
Choose a unit vector $\mathbf{u}$ at (x_0, y_0) and follow the line through (x_0, y_0) in that direction.

Slicing in Any Direction: Step by Step



Slice the surface with the vertical plane standing over that line — the same move as before, but the plane is no longer parallel to an axis.

Slicing in Any Direction: Step by Step



The slice is again a curve, and along it f is a function of the single arc-length variable s.

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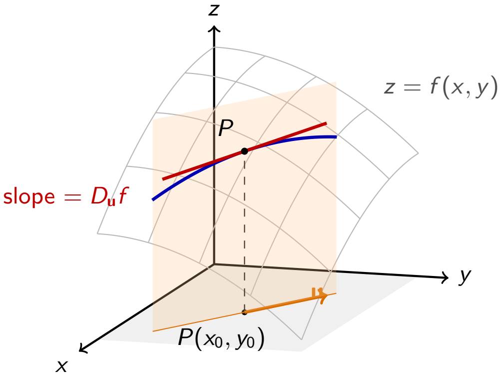

Its slope at P is the **directional derivative** $D_{\mathbf{u}}f$: the rate of change of f as we set off from P in the direction $\mathbf{u}$.

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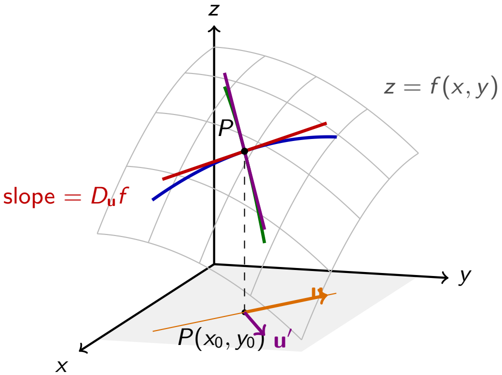

A different direction $\mathbf{u}'$ gives a different plane, a different trace and a different slope — here f was rising gently along $\mathbf{u}$ but falls steeply along $\mathbf{u}'$.

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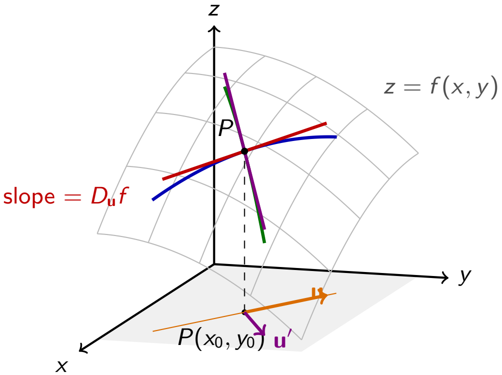

So each point carries *one rate per direction*. Taking $\mathbf{u} = \mathbf{i}$ and $\mathbf{u} = \mathbf{j}$ recovers f_x and f_y: the partials are two samples from this whole family.

A Formula, via the Chain Rule

Along the line $x = x_0 + su_1$, $y = y_0 + su_2$ we have $\dfrac{dx}{ds} = u_1$ and $\dfrac{dy}{ds} = u_2$.
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$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P} &= \left(\frac{\partial f}{\partial x}\right)_P \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_P \frac{dy}{ds} \\
&= \left(\frac{\partial f}{\partial x}\right)_P u_1 + \left(\frac{\partial f}{\partial y}\right)_P u_2 \\
&= \left[\left(\frac{\partial f}{\partial x}\right)_P \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_P \mathbf{j}\right] \cdot \left[u_1\mathbf{i} + u_2\mathbf{j}\right].
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So no new limit ever needs to be computed: the directional derivative in *every* direction is determined by the two partials. And the last line has split into a **dot product** of $\mathbf{u}$ with a vector built from f alone — which deserves a name.

The Gradient Vector

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The **gradient** of $f(x, y)$ at $P(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j},$$

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If $f(x, y)$ is differentiable in an open region containing $P(x_0, y_0)$, then for every unit vector $\mathbf{u}$,

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The gradient packages both partials into one *vector*, and the whole family of directional derivatives is recovered by dotting it with $\mathbf{u}$. Note the hypothesis once more: **differentiability**, which is what licensed the chain rule on the previous slide.

Worked Example

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- $\mathbf{v}$ is not a unit vector: $|\mathbf{v}| = \sqrt{9 + 16} = 5$, so $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$.

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$$f_x(2, 0) = 1, \qquad f_y(2, 0) = 2, \qquad \nabla f = \mathbf{i} + 2\mathbf{j}.$$

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Always normalise first. Using $\mathbf{v}$ itself would scale the answer by $|\mathbf{v}|$ and it would no longer be a rate per unit distance.

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Writing the dot product in terms of the angle θ between $\mathbf{u}$ and ∇f, and using $|\mathbf{u}| = 1$:

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So the gradient does not merely compute rates: its *direction* is the direction of steepest ascent, and its *length* is that steepest rate.

Worked Example: Steepest Ascent and Descent

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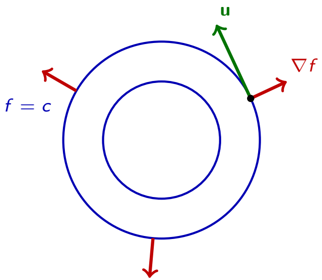
The surface is a bowl, and the level curve through $(1, 1)$ is a circle — whose tangent directions are exactly the zero-change ones.

The Gradient Is Perpendicular to the Level Curves



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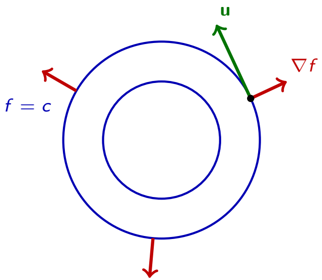

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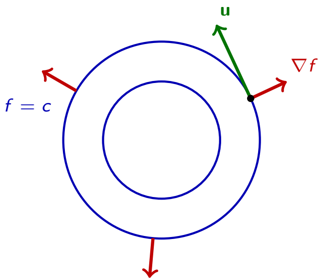

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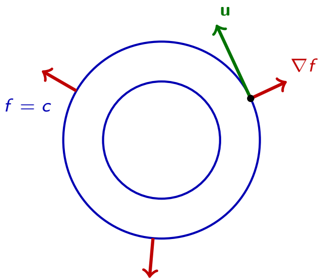

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Keep this fact in view. When we come to constrained optimisation, the constraint will be a level curve, and "the gradients are parallel there" will be the whole idea behind Lagrange multipliers.

Practice

- Find the directional derivative of $f(x, y) = x^2 + xy$ at $P(1, 2)$ in the direction of $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

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- For $f(x, y) = x^2 + y^2$, compute ∇f at $(1, 2)$ and verify directly that it is perpendicular to the tangent of the level curve through that point.
- Show that if $\nabla f = \mathbf{0}$ at P, then $D_{\mathbf{u}} f = 0$ for *every* direction $\mathbf{u}$. What kind of point is P then likely to be?

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- By the chain rule, $D_{\mathbf{u}}f = f_x u_1 + f_y u_2 = \nabla f \cdot \mathbf{u}$, where $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j}$ is the **gradient**. So two numbers determine the rate in every direction — and $\mathbf{u}$ must be a *unit* vector.

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- Since $D_{\mathbf{u}} f = |\nabla f| \cos\theta$: f rises fastest along ∇f at the rate $|\nabla f|$, falls fastest along $-\nabla f$, and does not change at all in directions orthogonal to ∇f.
- Consequently ∇f is **perpendicular to the level curves** of f — the fact behind Lagrange multipliers later in the course.