Lecture 09
Higher Partial Derivatives, Differentiability and the Chain Rule

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September 21, 2026

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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Outline

The Four Second-Order Partials

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Remark

Mind the order. In the subscript form f_{xy} we differentiate **left to right**: first in x, then in y. In the ∂ form the variables read **right to left**: in $\frac{\partial^2 f}{\partial y\,\partial x}$ the x nearest f comes first. The two notations agree.

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Example

For $f(x,y) = x\cos y + y\,e^x$: $f_x = \cos y + ye^x$ and $f_y = -x\sin y + e^x$, so

$$f_{xx} = y\,e^x, \quad f_{yy} = -x\cos y, \quad f_{xy} = -\sin y + e^x, \quad f_{yx} = -\sin y + e^x.$$

The Mixed Derivative Theorem

Theorem (Clairaut's theorem on equality of mixed partials)

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing (a, b) and all four are continuous at (a, b), then

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Example

For $w = xy + \dfrac{e^y}{y^2 + 1}$ the stated order needs the quotient rule. Reversing it, $\dfrac{\partial w}{\partial x} = y$, so $\dfrac{\partial^2 w}{\partial x\,\partial y} = 1$.

What Should "Differentiable" Mean?

- **One variable.** f is differentiable at x_0 exactly when $\Delta y = f'(x_0)\Delta x + \epsilon\,\Delta x$ with $\epsilon \to 0$: a *linear* function approximates f with an error small compared with Δx. (Here, we consider $y = f(x)$.)

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Theorem (Increment Theorem)

Suppose the first partials f_x and f_y of $f(x, y)$ are defined throughout an open region R containing (x_0, y_0), and f_x, f_y are continuous at (x_0, y_0). Then

$$\Delta z = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0, y_0)$$

satisfies

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y,$$

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The first two terms are *linear* in the increments; the last two vanish faster — the surface is well approximated near $P = (x_0, y_0)$ by a plane.

Differentiability, Defined

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$z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist and

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f_x, f_y continuous near (x_0, y_0) $\longrightarrow$ f differentiable at (x_0, y_0) $\longrightarrow$ f continuous at (x_0, y_0)

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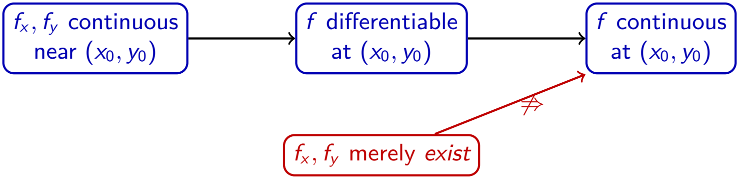

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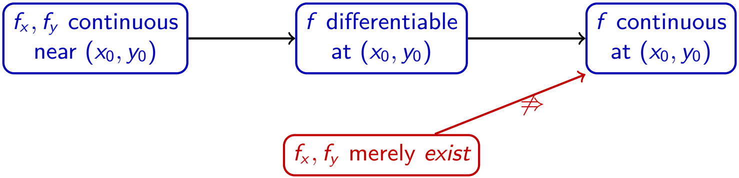

The left arrow is the Increment Theorem, and it is what we check in practice — the ϵ_1, ϵ_2 condition is almost never verified directly. The right arrow holds because letting $\Delta x, \Delta y \to 0$ makes every term vanish, so $\Delta z \to 0$. Neither arrow reverses.

The Chain Rule: One Independent, Two Intermediate Variables

Let $w = f(x, y)$ where *both* $x = x(t)$ and $y = y(t)$ depend on a single variable t. As t changes, $(x(t), y(t))$ traces a curve in the plane and w changes along it.

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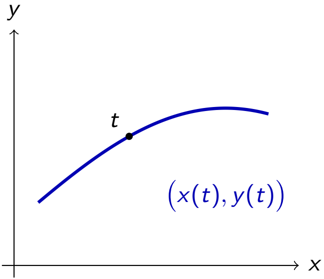
- t now reaches w by *two* routes, through x and through y, so the answer is a **sum** of two terms.
- Note the mixed symbols: ∂ for the derivatives of w in its two inputs, d for x and y, each depending on the single variable t.
- The hypothesis is **differentiability**, not just existence of the partials. The next slide shows exactly where that is used.

Where the Chain Rule Comes From: Step by Step



As t varies, the point $\big(x(t), y(t)\big)$ moves along a curve in the domain of f, and $w = f(x, y)$ changes along it.

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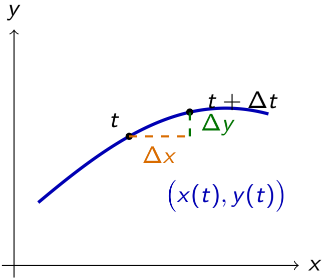

Advance t by Δt. The point moves by Δx in the x-direction *and* Δy in the y-direction — both change at once.

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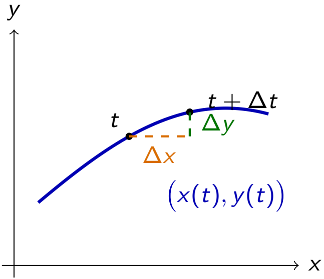

Because f is differentiable, the increment splitting applies:

$$\Delta w = f_x\,\Delta x + f_y\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.$$

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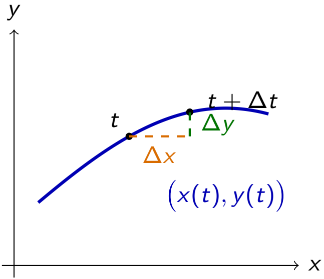

Divide by Δt: $$\frac{\Delta w}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

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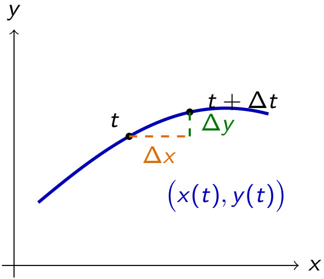

Let $\Delta t \to 0$. The ratios become $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and $\epsilon_1, \epsilon_2 \to 0$ kill the last two terms: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$.

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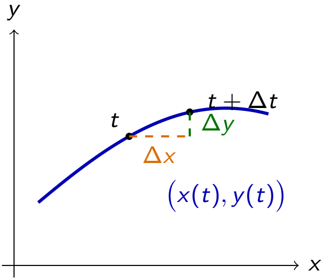

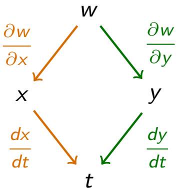

The same formula as a picture: t reaches w along *two* routes, one through x and one through y.

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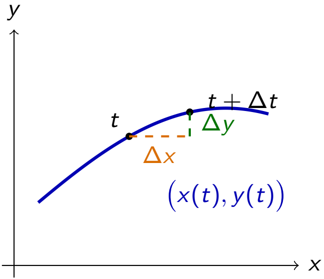

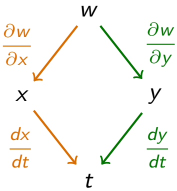

The recipe: multiply the derivatives along each branch from w down to t, then add over all the branches.

Worked Example: Moving Around a Circle

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Find $\dfrac{dw}{dt}$ for $w = xy$ along the path $x = \cos t$, $y = \sin t$, and evaluate it at $t = \pi/2$.

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Check by substituting first. As a function of t,

$w = \cos t \sin t = \frac{1}{2}\sin 2t$, so $\dfrac{dw}{dt} = \cos 2t$ — the same. Either way

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\pi = -1.$$

Substituting first is fine here, but it is often impossible in practice — which is what makes the chain rule worth having.

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More Variables, Same Recipe

- **Three intermediate variables.** If $w = f(x, y, z)$ with x, y, z differentiable in t,

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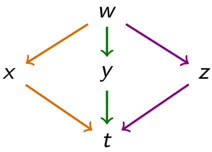
- **Two independent variables.** If x, y, z are functions of r and s, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

  and similarly for s: every d becomes a ∂, with the other independent variable held fixed.



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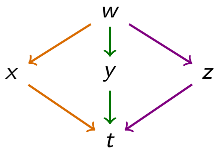

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The tree-diagram recipe: label each arrow with the corresponding derivative, then **multiply along each path** from w down to the independent variable and **add over all paths**.

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$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}, \qquad F_y \neq 0.}$$

Implicit Differentiation via the Chain Rule

Suppose $F(x, y) = 0$ defines y implicitly as a differentiable function of x. Apply the chain rule to $w = F(x, y)$ along the path $x = x$, $y = y(x)$:

$$\frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}.$$

But $w = F(x, y)$ is identically 0, so $dw/dx = 0$, giving

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}, \qquad F_y \neq 0.}$$

Example

For $y^2 - x^2 - \sin(xy) = 0$, take $F(x, y) = y^2 - x^2 - \sin(xy)$. Then $F_x = -2x - y\cos(xy)$ and $F_y = 2y - x\cos(xy)$, so

$$\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

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- Find $\dfrac{dw}{dt}$ if $w = \ln(x^2 + y^2)$, $x = e^t$, $y = e^{-t}$; and $\dfrac{dw}{dt}$ at $t = 0$ if $w = x\,e^y$, $x = t^2$, $y = 1 - t$.

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- If $w = x^2 + y^2$ with $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$, and explain the answers geometrically.

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- If $w = x^2 + y^2$ with $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$, and explain the answers geometrically.
- Use $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ for the folium $x^3 + y^3 = 6xy$.

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- The **chain rule** adds one term per intermediate variable, and follows directly from that increment splitting — which is why differentiability is the hypothesis.
- **Tree diagrams** reduce every version to one instruction: multiply along each path, add over the paths. A by-product is $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ for $F(x, y) = 0$.