Lecture 08
Functions of Several Variables: Limits, Continuity and Partial Derivatives

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Outline

From One Variable to Two

- A function of two variables takes a *point* of the plane and returns a real number,

$$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad w = f(x, y),$$

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- Since the domain now sits in the plane, we need a little vocabulary: (x_0, y_0) is an **interior point** of a region R if some disk around it lies entirely in R, and a **boundary point** if every disk around it meets both R and its complement. R is **open** if all its points are interior, **closed** if it contains its boundary.

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Example

$f(x, y) = \sqrt{9 - x^2 - y^2}$ has domain the closed disk $x^2 + y^2 \leq 9$ and range $[0, 3]$; its graph is a hemisphere and its level curves are circles.

The Limit of a Function of Two Variables

Definition

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ means: for every $\epsilon > 0$ there is a $\delta > 0$ such that for all (x,y) in the domain of f,

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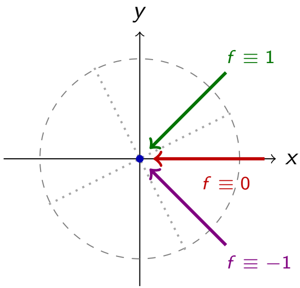
- What is genuinely new: on the line there were two ways in, from the left and from the right. In the plane there are infinitely many paths, and the definition demands the *same* L along every one of them.

The Two-Path Test



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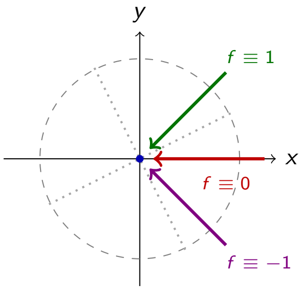

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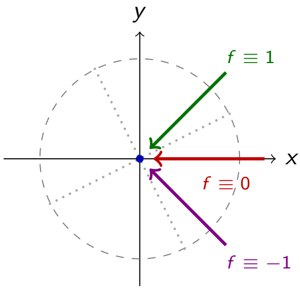

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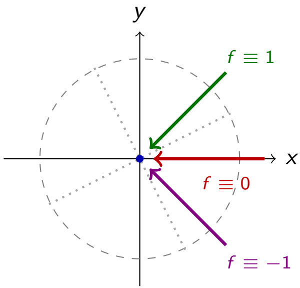

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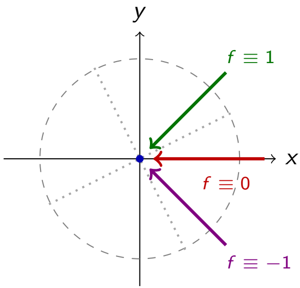

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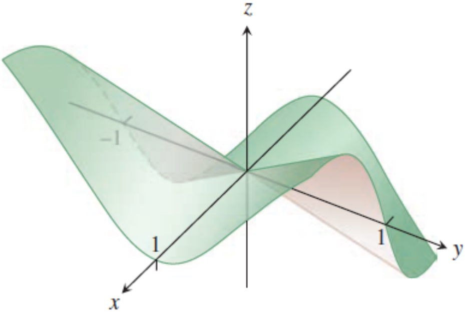
One-sided test: matching values along *some* paths never proves a limit exists — and straight lines alone are not enough, since curved paths count too.

Example - 1

Problem

Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.



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- First observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0. To see if this is true, we apply the definition of limit.
- Let $\epsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2+y^2} < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \delta$$

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- Since $y^2 \leq x^2 + y^2$ we have that
$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x| = 4\sqrt{x} \leq 4\sqrt{x^2+y^2}.$$
- So if we choose $\delta = \epsilon/4$ and let $0 < \sqrt{x^2+y^2} < \delta$, we get
$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < 4\sqrt{x^2+y^2} < 4\delta = 4(\tfrac{\epsilon}{4}) = \epsilon.$$
- It follows from the definition that
$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

Continuity

Definition

$f(x, y)$ is **continuous at** (x_0, y_0) if f is defined there, the limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and the two are equal. f is **continuous** if it is continuous at every point of its domain.

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$f(x, y) = \dfrac{2xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$, with $f(0, 0) = 0$, is continuous everywhere except the origin: away from 0 it is a rational function, and at 0 the limit does not exist, as we just saw. No choice of $f(0, 0)$ can repair it.

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Why?

Rates of Change in the Coordinate Directions

With two inputs there is no single rate of change — it depends on the direction we move in. The simplest thing that works is to move parallel to one axis at a time.

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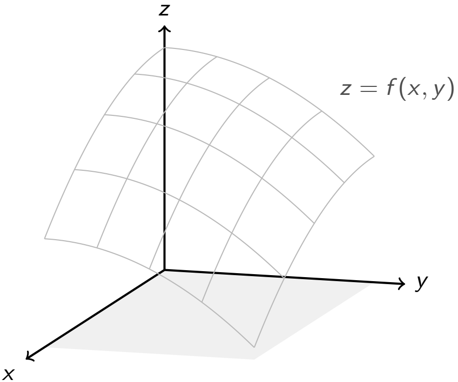
- Notation: $\dfrac{\partial f}{\partial x}$, f_x, $\dfrac{\partial z}{\partial x}$, z_x; values at a point as $f_x(x_0, y_0)$.

A Partial Derivative Is an Ordinary Slope: Step by Step



The surface $z = f(x, y)$ over a region of the xy-plane.

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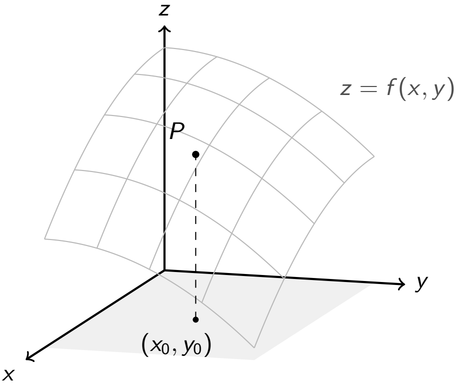

Fix (x_0, y_0) in the domain; above it sits $P\left(x_0, y_0, f(x_0, y_0)\right)$ on the surface.

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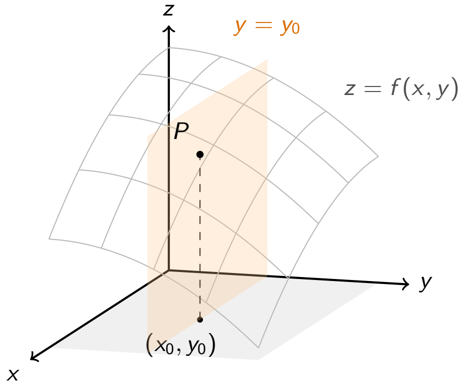

Freezing y at y_0 means slicing the surface with the vertical plane $y = y_0$.

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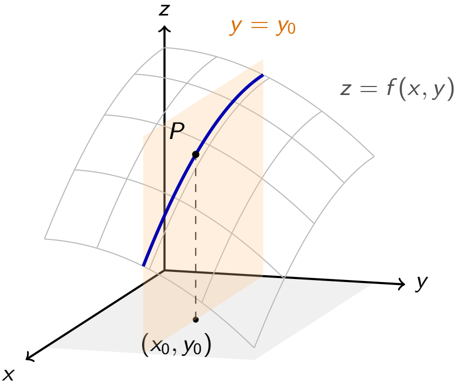

The slice is a curve $z = f(x, y_0)$ — an ordinary function of the single variable x.

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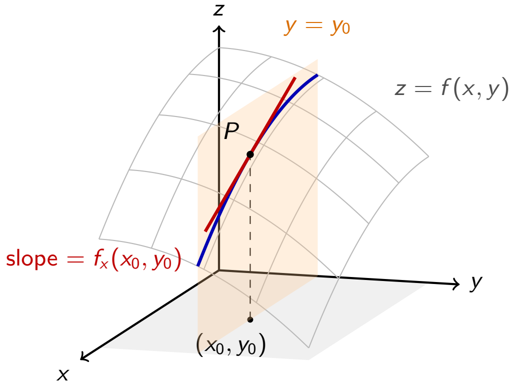

Its slope at P is an ordinary derivative, and that number is $f_x(x_0, y_0)$.

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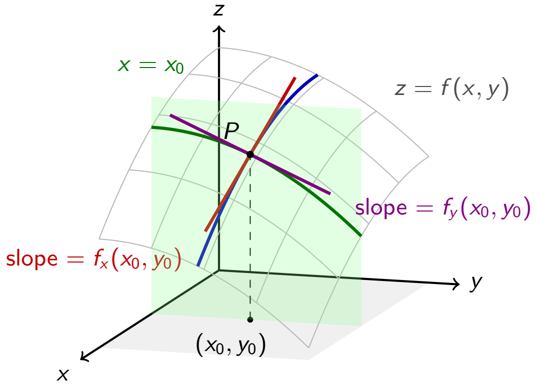

Freezing x at x_0 instead gives a second curve $z = f(x_0, y)$, whose slope at P is $f_y(x_0, y_0)$.

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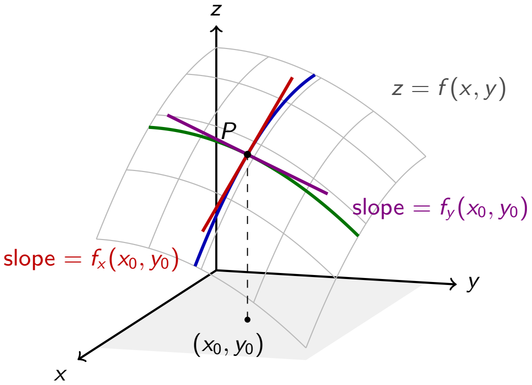

So at each point of the surface there are *two* tangent lines and two rates of change — one per coordinate direction.

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The working rule: to find f_x, treat y as a constant and differentiate in the ordinary way; for f_y, treat x as a constant. Every one-variable rule — sum, product, quotient, chain — still applies.

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Everything extends verbatim to three or more variables: differentiate with respect to one, hold *all* the others fixed. For $f(x, y, z) = x \sin(y + 3z)$, $f_z = 3x \cos(y + 3z)$.

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Exactly the one-variable implicit method — the only new instruction is "hold the other independent variable fixed".

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Remark

In one variable, differentiability at a point forces continuity there. Partial derivatives are weaker: each sees only *one* line through the point and is blind to what happens off those two lines. Next lecture we introduce the stronger condition — **differentiability** — that does imply continuity.

Practice

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- Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^2 + y^2 + z^2 = 9$ defines z implicitly.
- The plane $y = 1$ cuts the surface $z = x^2 - xy$ in a curve. Find the slope of its tangent at $(2, 1, 2)$.

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- **Warning:** both partials can exist at a point where f is not continuous — existence of partials is not the right generalisation of "differentiable".