Lecture 07
Improper Integrals

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Outline

Improper integrals of Type II: Finite Intervals

The integral of a discontinuous function

If the function $f(x)$ is discontinuous at the point $x = a$ of the interval $[a, c]$, then define

$$\int_a^c f(x)dx = \lim_{b \to a^+} \int_b^c f(x)dx.$$

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If the function $f(x)$ is discontinuous at some point $x = x_0$ inside the interval $[a, c]$, then define

$$\int_a^c f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx,$$

if both the improper integrals on the right-hand side of the equation exist.

Problem

Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

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Solution

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{1-x}} &= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} \\
&= -\lim_{b\to 1^-} \left[2\sqrt{1-x}\right]_0^b \\
&= -\lim_{b\to 1^-} \left[2(\sqrt{1-b}-1)\right] \\
&= -(-2) = 2
\end{aligned}$$

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- Therefore,

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{a \to 0^-} \int_{-1}^{a} \frac{dx}{x^2} + \lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{x^2}.$$

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- Now, $\lim_{a \to 0^+} \int_{-1}^{a} \frac{dx}{x^2} = \lim_{a \to 0^-} \left[-\frac{1}{x} \right]_{-1}^{a} = \lim_{a \to 0^-} \left[-\frac{1}{a} + 1 \right]$, and $\lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{x^2} = \lim_{b \to 0^+} \left[-\frac{1}{x} \right]_{b}^{1} = \lim_{b \to 0^+} \left[-1 + \frac{1}{b} \right] = \infty$.

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- Therefore, the integral diverges in the interval $(-1, 1)$.

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CAVEAT: $\left[\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x} \right]_{-1}^{1} = -2 \right]$

Remark

If the function $f(x)$ defined on the interval $[a, b]$, has a finite number of points of discontinuity in this interval, then the integral

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If the function $f(x)$ defined on the interval $[a, b]$, has a finite number of points of discontinuity in this interval, then the integral is defined as follows:

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \ldots + \int_{a_n}^{b} f(x)dx,$$

if each of the integrals in the RHS converges. If one of the integrals in RHS is divergent, then $\int_a^b f(x)dx$ is called divergent.

Comparision Theorem with discontinuity

Theorem (Comparision Theorem with a discontinuity)

Let $f(x)$ and $g(x)$ be discontinuous at the point c on the interval $[a, c]$, and at all other points of this interval $0 \leq f(x) \leq g(x)$ holds.

1. *If $\int_a^c g(x)dx$ converges, then $\int_a^c f(x)dx$ converges.*
2. *If $\int_a^c f(x)dx$ diverges, then $\int_a^c g(x)dx$ diverges.*

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2. *If $\int_a^c f(x)dx$ diverges, then $\int_a^c g(x)dx$ diverges.*

Theorem (Absolute convergence with a discontinuity)

If $f(x)$ is discontinuous at the point c and the improper integral $\int_a^c |f(x)|dx$ converges, then the integral $\int_a^c f(x)dx$ of the function itself converges.

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Solution

- The integrand is discontinuous at $x = 0$.
- Now, $\frac{1}{\sqrt{x}+4x^3} < \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{1}{\sqrt{x}}dx$ exists.
- Therefore, $\int_0^1 \frac{1}{\sqrt{x}+4x^3}dx$ exists.

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- Therefore, $\int_0^1 \frac{1}{\sqrt{x}+4x^3} dx$ exists.

Problem

1. Verify that the integral $\int_a^c \frac{1}{(c-x)^\alpha} dx$ converges for $\alpha < 1$ and diverges for $\alpha \geq 1$.
2. Verify that the integral $\int_a^c \frac{1}{(x-a)^\alpha} dx$ converges for $\alpha < 1$ and diverges for $\alpha \geq 1$.