Lecture 06
Improper Integrals

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September 10, 2026

Outline

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- Both are called **improper integrals**; we handle them by integrating up to (or away from) the trouble spot, and then taking a limit.
- Today: Type I, in detail, along with a tool to decide convergence without always evaluating the integral exactly. Type II follows next lecture.

Infinite Limits of Integration

Definition

If f is continuous on $[a, \infty)$ and $\lim_{b \to \infty} \int_a^b f(x)\, dx$ exists and is finite, we define

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx,$$

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Similarly, for the other unbounded intervals:

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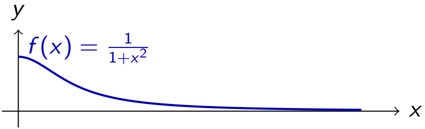
Similarly, for the other unbounded intervals:

- $\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx.$
- $\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ for any fixed c — and **both** pieces must converge.

From a Finite Area to an Improper Integral: Step by Step



The curve $y = f(x) = \dfrac{1}{1+x^2}$ on $[0, \infty)$.

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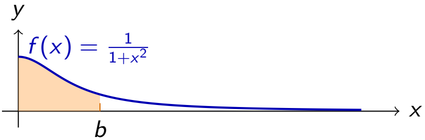

The area over $[0, b]$ is $\int_0^b f(x)\,dx$ — an ordinary, finite integral.

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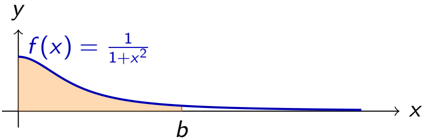

As b grows, so does the shaded area — but it seems to be filling up, not blowing up.

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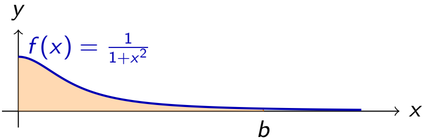

Even for large b, the area keeps growing more and more slowly.

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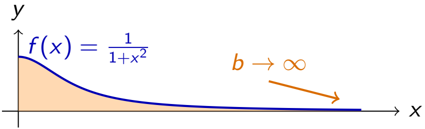

Let $b \to \infty$: the shaded region becomes the whole (unbounded) area under the curve.

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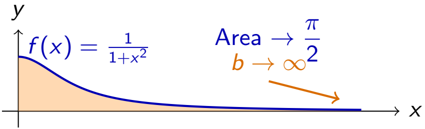

The limit $\lim_{b\to\infty} \int_0^b f(x)\,dx$ exists and is finite — this defines $\int_0^\infty f(x)\,dx$.

Worked Example

Problem

Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$.

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$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}\\ &= \lim_{b\to\infty}\Big[\tan^{-1} x\Big]_0^b\\ &= \lim_{b\to\infty} \tan^{-1} b\\ &= \frac{\pi}{2}.\end{aligned}$$

The improper integral converges, and equals $\pi/2$ — matching the picture on the previous slide.

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The p-Integral: A Key Benchmark

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$$\boxed{\int_1^\infty \frac{dx}{x^p} \text{ converges} \iff p > 1}$$

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- The second piece is $\pi/2$, from the previous example. For the first,

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Both pieces had to converge separately — this is not just $[\tan^{-1} x]_{-\infty}^{\infty}$ evaluated carelessly in one step.

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2. *If* $\int_a^\infty f(x)\,dx$ *diverges, so does* $\int_a^\infty g(x)\,dx$*.*

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Note we never evaluated the original integral — comparison alone settled convergence.

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Then $\int_a^\infty f(x) g(x)\, dx$ converges.

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Then $\displaystyle\int_a^\infty f(x)g(x)\,dx$ converges.

Idea: f decaying to 0 tames the oscillation of g's (bounded) running integral — a one-sided analogue of the alternating series test.

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Solution

- Take $f(x) = \dfrac{1}{x}$ (decreasing, $\to 0$) and $g(x) = \sin x$ (continuous, with $\left|\int_1^t \sin x\, dx\right| = |\cos 1 - \cos t| \leq 2$).

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- By Dirichlet's Test, $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ **converges**.
- It does *not* converge **absolutely**: comparing $\int_1^\infty \left|\frac{\sin x}{x}\right| dx$ block-by-block over each interval $[2\pi n, 2\pi(n+1)]$ bounds it below by a constant multiple of the divergent series $\sum \frac{1}{n+1}$.

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So $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ is **conditionally convergent** — it converges, but not absolutely.

By the Dirichlet test, the improper integral converges. But it is not absolutely convergent. For,

$$\begin{aligned}
\int_1^{2\pi k} \left|\frac{\sin x}{x}\right| \, dx &\geq \sum_{n=1}^{k-1} \int_{2\pi n}^{2\pi(n+1)} \left|\frac{\sin x}{x}\right| \, dx \\
&\geq \sum_{n=1}^{k-1} \frac{1}{2\pi(n+1)} \int_{2\pi n}^{2\pi(n+1)} |\sin x| \, dx \\
&= \sum_{n=1}^{k-1} \frac{1}{2\pi(n+1)} \int_0^{2\pi} |\sin x| \, dx \\
&= \sum_{n=1}^{k-1} \frac{2}{\pi(n+1)}.
\end{aligned}$$

As $\sum_{n=1}^{\infty} \frac{2}{\pi(n+1)}$ is diverges, hence the improper integral diverges. That is, the improper integral $\int_1^{+\infty} \frac{\sin x}{x} dx$ converges, but not absolutely!

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- Determine whether $\int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$ converges or diverges, using the p-integral as a benchmark together with the Direct Comparison Test.
- Evaluate $\int_{-\infty}^0 xe^x\,dx$ (integrate by parts first).

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- **Dirichlet's Test** handles oscillating integrands like $\sin x/x$ that Direct Comparison can't reach — and can give **conditional** convergence (converges, but not absolutely).
- Next: improper integrals where the **integrand** (not the interval) is the source of the trouble.