Lecture 05
Areas of Surfaces of Revolution

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Outline

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- We will derive a formula the same way as before: approximate by simple pieces, then take a limit.

Building Block 1: A Straight Segment Gives a Cylinder

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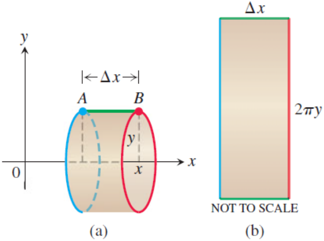
- Unrolled, its lateral surface is a rectangle of sides Δx and $2\pi y$, so its area is

$$2\pi y\,\Delta x.$$



Building Block 2: A Slanted Segment Gives a Frustum

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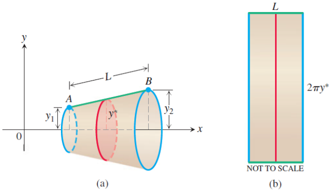
- Revolving AB about the x-axis sweeps out a **frustum of a cone**.
- Unrolled, its lateral surface is (again) a rectangle, of sides L and $2\pi y^*$, where $y^* = \dfrac{y_1 + y_2}{2}$ is the **average height**. So its area is

$$2\pi y^* L.$$



The General Curve

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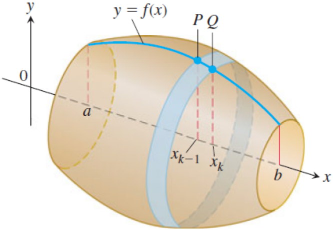
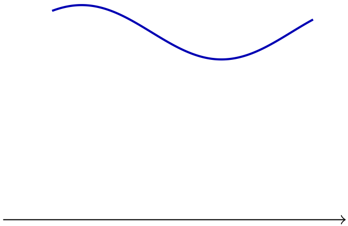
- The arc PQ sweeps out a curved band of the surface — we approximate its area using the two building blocks above.



From a Curve to a Surface: Step by Step



The curve $y = f(x) \geq 0$ on $[a, b]$, about to be revolved about the x-axis.

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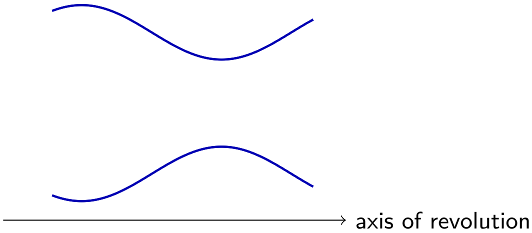

Revolving traces out a surface — a thin skin, not a solid.

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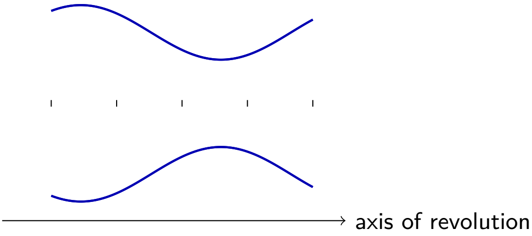
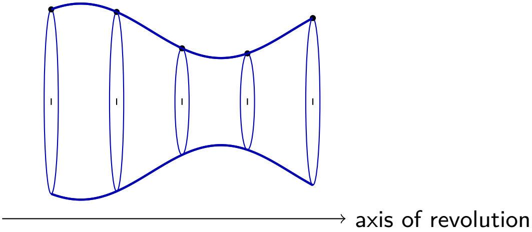


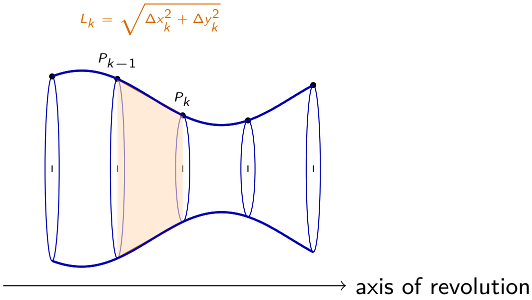
Partition $[a, b]$: mark the circular rims at each x_k.

From a Curve to a Surface: Step by Step



Each rim is a circle of radius $f(x_k)$, swept out by the point $P_k = (x_k, f(x_k))$.

From a Curve to a Surface: Step by Step



Approximate the band swept by arc $P_{k-1}P_k$ by the frustum swept by the *chord*: slant length $L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$.

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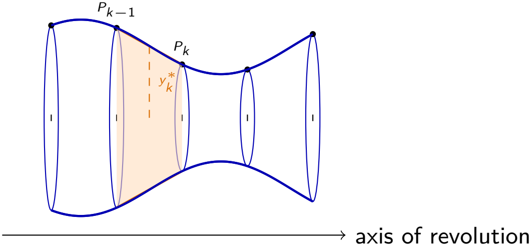

Frustum area $= 2\pi y_k^* L_k$, average radius $y_k^* = \frac{1}{2}\big(f(x_{k-1}) + f(x_k)\big)$.

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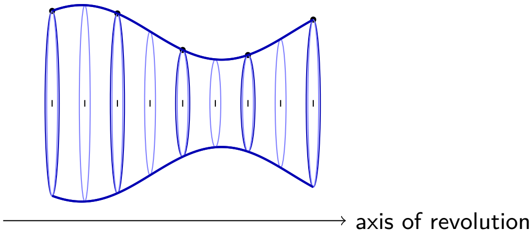

Refine the partition: more, thinner frustum bands hug the true surface more closely.

Zooming In: One Frustum Band

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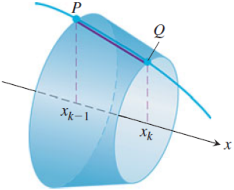
- This is exactly Building Block 2 above, applied locally to one piece of the partition.



Formula for a Single Band

- The frustum swept out by chord $P_{k-1}P_k$ has average radius

$$y_k^* = \frac{f(x_{k-1}) + f(x_k)}{2}$$

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- Summing over all the bands, the surface area is approximately

$$\sum_{k=1}^{n} \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

The Mean Value Theorem and the Limit

- By the **Mean Value Theorem**, there is a point $c_k \in (x_{k-1}, x_k)$ with $f'(c_k) = \dfrac{\Delta y_k}{\Delta x_k}$, i.e. $\Delta y_k = f'(c_k)\,\Delta x_k$.

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- As the partition is refined, $f(x_{k-1}) + f(x_k) \to 2f(c_k)$, and the sum becomes a Riemann sum for $2\pi f(x)\sqrt{1+f'(x)^2}$. Letting $\|P\| \to 0$,

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{1 + f'(c_k)^2}\,\Delta x_k \\ = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\;dx.\end{aligned}$$

Area of a Surface of Revolution

Definition

If f' is continuous and $f \geq 0$ on $[a, b]$, the area of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

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- If f can be negative, use $|f(x)|$ in place of $f(x)$ — a radius is never negative.

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$$S = \int_1^2 2\pi\,(2\sqrt{x})\,\sqrt{\frac{x+1}{x}}\;dx = \int_1^2 4\pi\,\sqrt{x}\cdot\frac{\sqrt{x+1}}{\sqrt{x}}\;dx$$

$$= \int_1^2 4\pi\sqrt{x+1}\;dx = 4\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_1^2$$

$$S = \frac{8\pi}{3}\left(3^{3/2} - 2^{3/2}\right) = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

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- Using the $x = g(y)$ formula, find the area of the surface generated by revolving $x = \sqrt{y}$, $1 \leq y \leq 4$, about the y-axis.

Summary

- Approximate the surface swept by a curve $y = f(x)$ using frustum bands swept by chords; the band over $[x_{k-1}, x_k]$ has area $2\pi y_k^* L_k$, with average radius y_k^* and slant length L_k.

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