Lecture 04
Arc Length

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Outline

Recap : Mean Value Theorem

Length of a Curve

Deriving the Formula

Examples

Mean Value Theorem

Theorem

Let f be a continuous function over a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is an element $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f^{'}(c).$$

Let $f(x) = x^2$ on the interval $[1, 3]$. By the Mean Value Theorem, there exists $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

Since

$$f'(x) = 2x,$$

we obtain

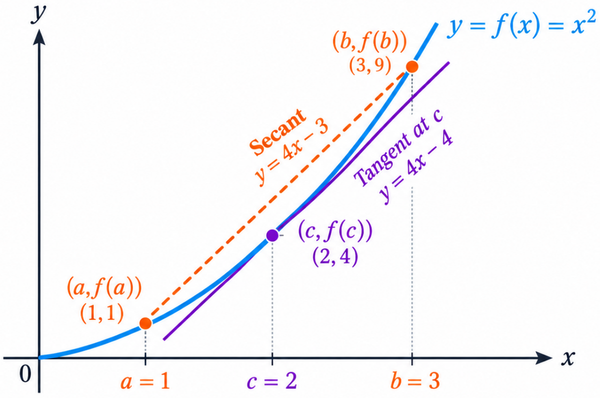
$$2c = \frac{9 - 1}{2} = 4,$$

and hence

$$\boxed{c = 2}.$$

Thus, the tangent to $y = x^2$ at $x = 2$ has the same slope as the secant line joining $(1, 1)$ and $(3, 9)$.

Mean Value Theorem



$$\underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{secant slope}} = \frac{9-1}{3-1} = 4 = \underbrace{f'(c)}_{\text{tangent slope}} \qquad \text{(parallel lines)}$$

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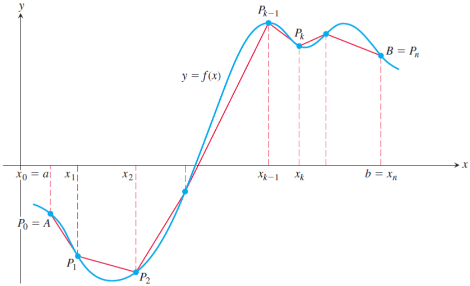
- The curve is the graph of a function $f : [a, b] \to \mathbb{R}$.
- A function f is called **smooth** on $[a, b]$ if f' is continuous there.
- We will derive a formula for the length of a smooth curve, by approximating it with straight-line segments and taking a limit — the same Riemann-sum idea used for area and volume.

A Curve and Its Polygonal Approximation



- Partition $[a, b]$: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and let $P_k = (x_k, f(x_k))$ be the corresponding point on the curve.

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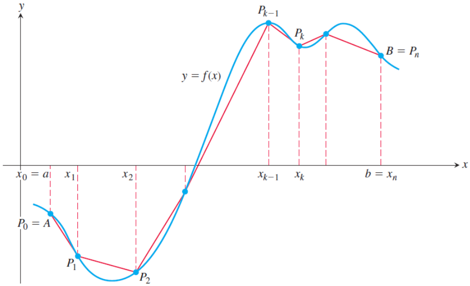
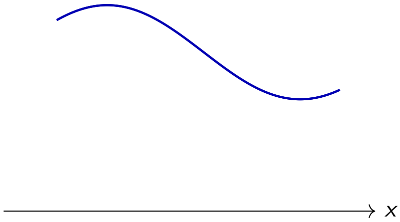


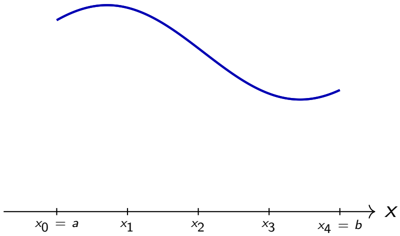
- Partition $[a, b]$: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and let $P_k = (x_k, f(x_k))$ be the corresponding point on the curve.
- Connecting consecutive points P_{k-1} and P_k by straight segments gives a **polygonal path** whose length approximates the length of the curve.

From a Curve to Its Length: Step by Step



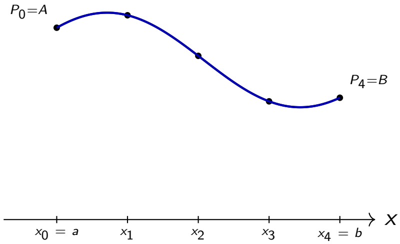
The curve $y = f(x)$ on $[a, b]$.

From a Curve to Its Length: Step by Step



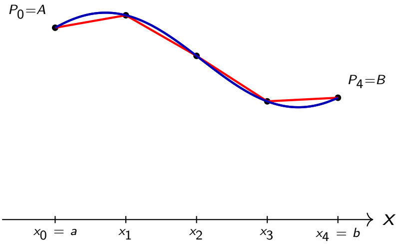
Partition $[a, b]$ into n sub-intervals.

From a Curve to Its Length: Step by Step



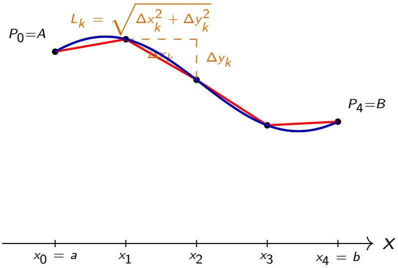
Mark the points $P_k = (x_k, f(x_k))$ on the curve.

From a Curve to Its Length: Step by Step



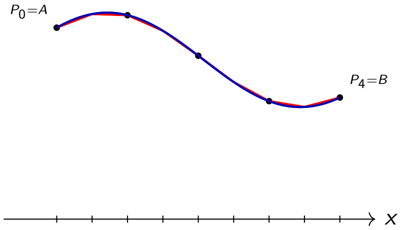
Connect consecutive points by straight chords: a polygonal path.

From a Curve to Its Length: Step by Step



One chord's length: $L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$.

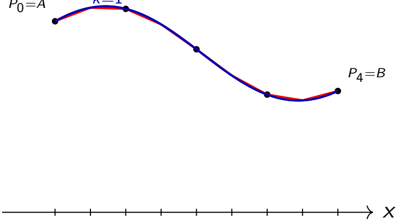
From a Curve to Its Length: Step by Step



Refine the partition: more, shorter chords hug the curve more closely.

From a Curve to Its Length: Step by Step

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} L_k = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$



As $n \to \infty$: exact arc length $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$.

Deriving the Length Formula

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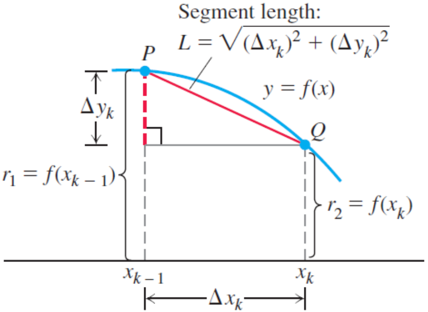
- The polygonal path's total length approximates the length of the curve:

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Zooming In: The Chord Length



- Here r_1 and r_2 just denote the curve's heights $f(x_{k-1})$ and $f(x_k)$ — the figure is drawn generally, but for us the curve is $y = f(x)$ throughout.

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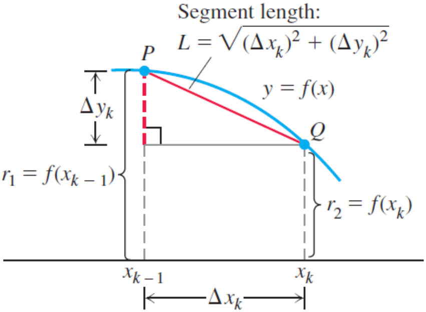

- Here r_1 and r_2 just denote the curve's heights $f(x_{k-1})$ and $f(x_k)$ — the figure is drawn generally, but for us the curve is $y = f(x)$ throughout.
- By the Pythagorean theorem, $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$ — exactly the chord length used above.

Deriving the Length Formula — contd.

- By the **Mean Value Theorem**, there is a point $c_k \in (x_{k-1}, x_k)$ such that $\Delta y_k = f'(c_k)\,\Delta x_k$.

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- Substituting,

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + f'(c_k)^2(\Delta x_k)^2} = \sum_{k=1}^{n} \Delta x_k \sqrt{1 + f'(c_k)^2}.$$

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- This is a Riemann sum for $\sqrt{1 + f'(x)^2}$. As the partition is refined ($\|P\| \to 0$),

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x_k \sqrt{1 + f'(c_k)^2} = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Length of a Curve

Definition

If f' is continuous on $[a, b]$, the **length (arc length)** of the curve $y = f(x)$ from $A = (a, f(a))$ to $B = (b, f(b))$ is

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- For a curve $x = g(y)$, $c \leq y \leq d$, swapping the roles of x and y gives

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Worked Example

Example

Find the length of the curve $y = \dfrac{x^3}{12} + \dfrac{1}{x}, \quad 1 \le x \le 4.$

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$f'(x) = \dfrac{x^2}{4} - \dfrac{1}{x^2}.$

$$\begin{aligned} 1 + f'(x)^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2. \end{aligned}$$

So $\sqrt{1 + f'(x)^2} = \dfrac{x^2}{4} + \dfrac{1}{x^2}$ (positive on $[1, 4]$), and

$$L = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \frac{61}{12} - \left(-\frac{11}{12}\right) = 6.$$

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Practice

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- Find the length of $y = \ln(\cos x)$ on $\left[0, \frac{\pi}{4}\right]$.
- Using the $x = g(y)$ formula, find the length of $x = \frac{1}{3}(y^2 + 2)^{3/2}$, $0 \leq y \leq 3$.

Summary

- Approximate a curve $y = f(x)$ by a polygonal path of chords; the chord over $[x_{k-1}, x_k]$ has length $L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$.

Summary

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- By the Mean Value Theorem, $\displaystyle\sum_k L_k \to \int_a^b \sqrt{1 + f'(x)^2}\, dx$ as the partition is refined — this is the **arc length** of the curve.

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- By the Mean Value Theorem, $\sum_k L_k \to \int_a^b \sqrt{1 + f'(x)^2}\,dx$ as the partition is refined — this is the **arc length** of the curve.
- For a curve $x = g(y)$: $L = \int_c^d \sqrt{1 + g'(y)^2}\,dy$, by the same argument with x and y swapped.