Lecture 03
Volumes by Slicing and Revolution

M. Rajesh Kannan and D. Sukumar

Department of Mathematics,
Indian Institute of Technology Hyderabad,
email: rajeshkannan@math.iith.ac.in, suku@math.iith.ac.in

September 4, 2026

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Outline

Recap: From Area to Volume

- Recall: for $f \geq 0$ on $[a, b]$, slicing the region under $y = f(x)$ into thin vertical strips of width Δx_k and summing $f(c_k)\Delta x_k$ gave us

$$A = \int_a^b f(x)\,dx.$$

Recap: From Area to Volume

- Recall: for $f \geq 0$ on $[a, b]$, slicing the region under $y = f(x)$ into thin vertical strips of width Δx_k and summing $f(c_k)\Delta x_k$ gave us

$$A = \int_a^b f(x)\,dx.$$

- **Today's question:** what if, instead of a flat region, we have a solid three-dimensional object — how do we find its volume?

Recap: From Area to Volume

- Recall: for $f \geq 0$ on $[a, b]$, slicing the region under $y = f(x)$ into thin vertical strips of width Δx_k and summing $f(c_k)\Delta x_k$ gave us

$$A = \int_a^b f(x)\, dx.$$

- **Today's question:** what if, instead of a flat region, we have a solid three-dimensional object — how do we find its volume?
- **Idea:** slice the *solid* instead of the region, and add up the volumes of the (thin) slices.

Volume by Slicing

- Partition $[a, b]$ using $a = x_0 < x_1 < \cdots < x_n = b$, and slice the solid by planes perpendicular to the x-axis at each x_k.

Volume by Slicing

- Partition $[a, b]$ using $a = x_0 < x_1 < \cdots < x_n = b$, and slice the solid by planes perpendicular to the x-axis at each x_k.
- Let $A(x)$ be the cross-sectional area of the solid at the point x. The volume of the thin slab between x_{k-1} and x_k is approximately $A(x_k)\,\Delta x_k$.

Volume by Slicing

- Partition $[a, b]$ using $a = x_0 < x_1 < \cdots < x_n = b$, and slice the solid by planes perpendicular to the x-axis at each x_k.
- Let $A(x)$ be the cross-sectional area of the solid at the point x. The volume of the thin slab between x_{k-1} and x_k is approximately $A(x_k)\,\Delta x_k$.
- Summing over all slabs gives the Riemann sum $\sum_{k=1}^{n} A(x_k)\,\Delta x_k$; letting $\|P\| \to 0$,

$$\text{Volume} = \int_a^b A(x)\,dx.$$

Working Rule & Example

1. Sketch the solid and identify a typical cross-section.

Working Rule & Example

1. Sketch the solid and identify a typical cross-section.
2. Find a formula for $A(x)$, the area of that cross-section.

Working Rule & Example

1. Sketch the solid and identify a typical cross-section.
2. Find a formula for $A(x)$, the area of that cross-section.
3. Find the limits of integration a, b along the slicing axis.

Working Rule & Example

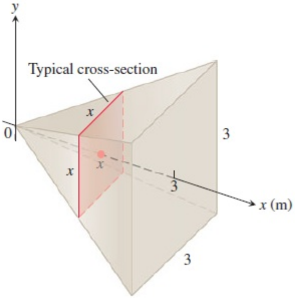
1. Sketch the solid and identify a typical cross-section.
2. Find a formula for $A(x)$, the area of that cross-section.
3. Find the limits of integration a, b along the slicing axis.
4. Integrate: Volume $= \int_a^b A(x)\,dx$.

Example

A pyramid 3 m high has a square base 3 m on a side. The cross-section perpendicular to the altitude, x meters down from the vertex, is a square of side x meters. Find its volume.

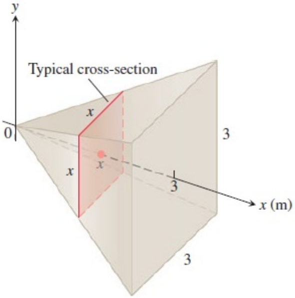
Example

A pyramid 3 m high has a square base 3 m on a side. The cross-section perpendicular to the altitude, x meters down from the vertex, is a square of side x meters. Find its volume.



Example

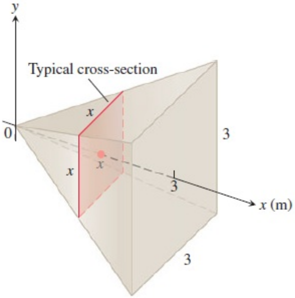
A pyramid 3 m high has a square base 3 m on a side. The cross-section perpendicular to the altitude, x meters down from the vertex, is a square of side x meters. Find its volume.



- $A(x) = x^2$, with x ranging from 0 to 3.

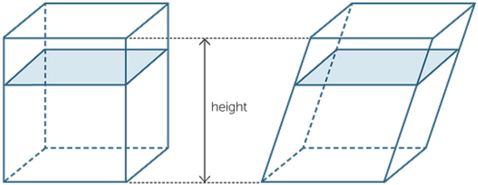
Example

A pyramid 3 m high has a square base 3 m on a side. The cross-section perpendicular to the altitude, x meters down from the vertex, is a square of side x meters. Find its volume.

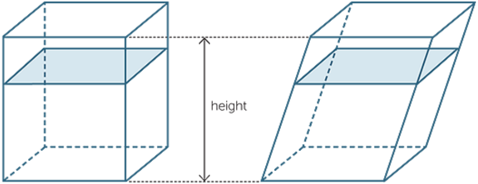


- $A(x) = x^2$, with x ranging from 0 to 3.
-
$$\text{Volume} = \int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = 9 \text{ m}^3.$$

Cavalieri's Principle

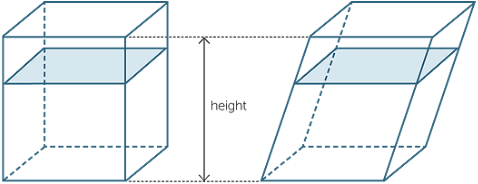


Cavalieri's Principle



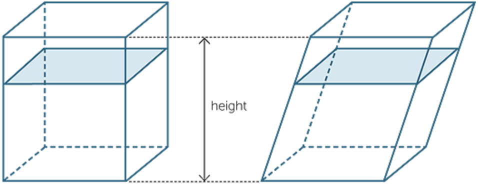
- **Cavalieri's Theorem.** Solids with the same altitude and the same cross-sectional area at every level have the same volume — immediate from Volume $= \int_a^b A(x)\,dx$: only $A(x)$ matters, not the shape of the solid.

Cavalieri's Principle

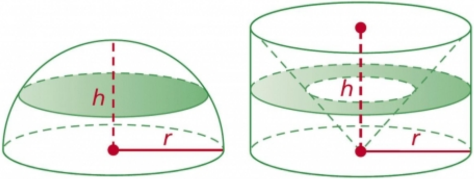


- **Cavalieri's Theorem.** Solids with the same altitude and the same cross-sectional area at every level have the same volume — immediate from Volume $= \int_a^b A(x)\,dx$: only $A(x)$ matters, not the shape of the solid.
- Example: a hemisphere of radius r has the same volume as a cylinder of radius/height r with a cone of radius/height r removed — both have cross-sectional area $\pi(r^2 - x^2)$ at height x.

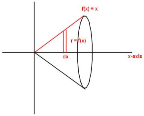
Cavalieri's Principle



- **Cavalieri's Theorem.** Solids with the same altitude and the same cross-sectional area at every level have the same volume — immediate from Volume $= \int_a^b A(x)\,dx$: only $A(x)$ matters, not the shape of the solid.
- Example: a hemisphere of radius r has the same volume as a cylinder of radius/height r with a cone of radius/height r removed — both have cross-sectional area $\pi(r^2 - x^2)$ at height x.



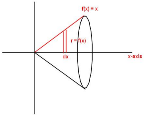
Solids of Revolution: Disk & Washer Methods



- **Disk method.** If the region under $y = R(x) \geq 0$ on $[a, b]$ is revolved about the x-axis, each cross-section is a disk of radius $R(x)$, so $A(x) = \pi R(x)^2$ and

$$\text{Volume} = \int_a^b \pi\, R(x)^2\, dx.$$

Solids of Revolution: Disk & Washer Methods



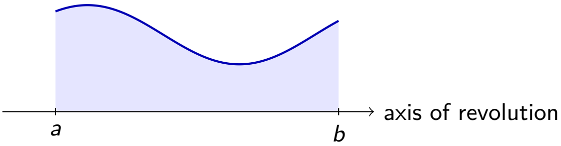
- **Disk method.** If the region under $y = R(x) \geq 0$ on $[a, b]$ is revolved about the x-axis, each cross-section is a disk of radius $R(x)$, so $A(x) = \pi R(x)^2$ and

$$\text{Volume} = \int_a^b \pi\, R(x)^2\, dx.$$

- Revolving about the y-axis instead (region between $x = R(y)$ and the y-axis, $c \leq y \leq d$):

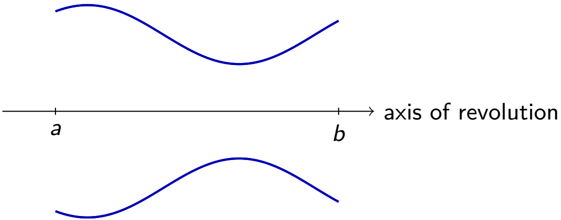
$$\text{Volume} = \int_c^d \pi\, R(y)^2\, dy.$$

Disk & Washer Methods: Step by Step



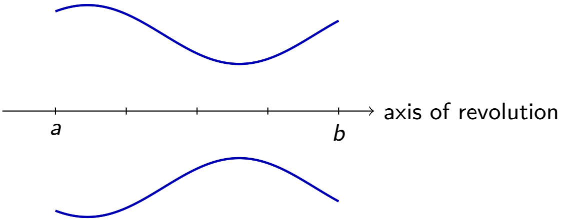
The region under $y = R(x) \geq 0$ on $[a, b]$, about to be revolved about the x-axis.

Disk & Washer Methods: Step by Step



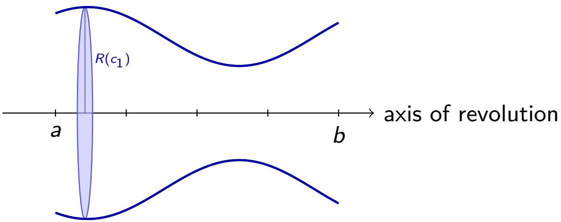
Revolving traces out a solid of revolution.

Disk & Washer Methods: Step by Step



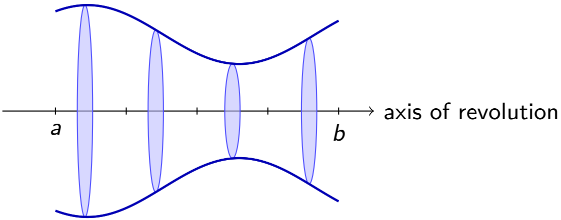
Partition $[a, b]$: slice the solid at each x_k.

Disk & Washer Methods: Step by Step



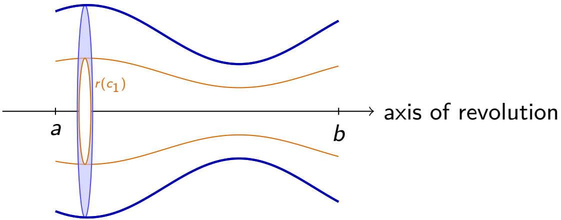
At $x = c_1$, the cross-section is a disk of radius $R(c_1)$: area $\pi R(c_1)^2$.

Disk & Washer Methods: Step by Step



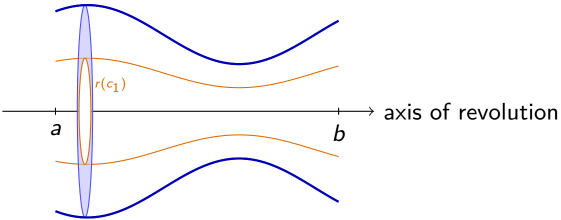
Disk method: Volume $\approx \sum_k \pi R(c_k)^2 \Delta x_k$.

Disk & Washer Methods: Step by Step

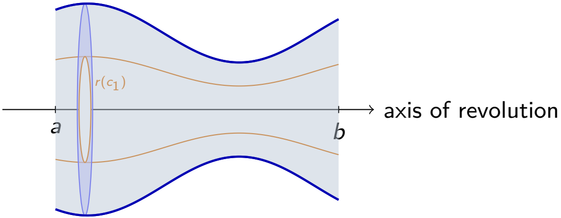


Washer method: with an inner boundary $r(x)$, each slice is an annulus of area $\pi\left[R(c_k)^2 - r(c_k)^2\right]$.

Disk & Washer Methods: Step by Step



Disk & Washer Methods: Step by Step



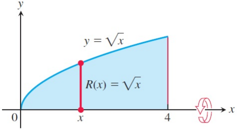
As $\|P\| \to 0$: disk $\text{Volume} = \displaystyle\int_a^b \pi R(x)^2\,dx$; washer

$$\text{Volume} = \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx.$$

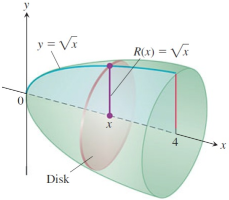
Worked Examples

Disk method. The region under $y = \sqrt{x}$ on $[0, 4]$ is revolved about the x-axis. Find the volume.

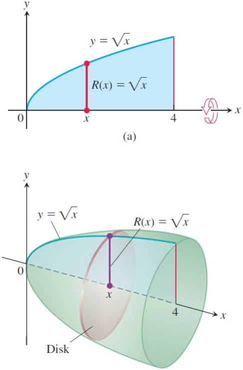
Disk method. The region under $y = \sqrt{x}$ on $[0, 4]$ is revolved about the x-axis. Find the volume.



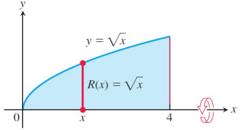
(a)



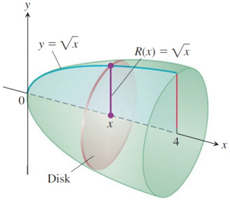
Disk method. The region under $y = \sqrt{x}$ on $[0, 4]$ is revolved about the x-axis. Find the volume.



Disk method. The region under $y = \sqrt{x}$ on $[0, 4]$ is revolved about the x-axis. Find the volume.



(a)



- $R(x) = \sqrt{x}$, so

$$\text{Volume} = \int_0^4 \pi x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi.$$

Washer method

Washer method. If the region is bounded between an outer radius $R(x)$ and an inner radius $r(x)$ (a hole down the middle, like a stack of washers), each cross-section is an annulus:

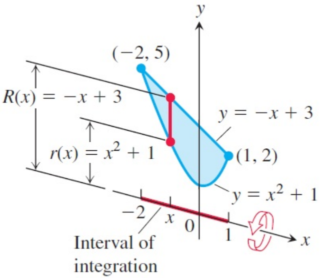
$$\text{Volume} = \int_a^b \pi \left[R(x)^2 - r(x)^2 \right] dx.$$

Example

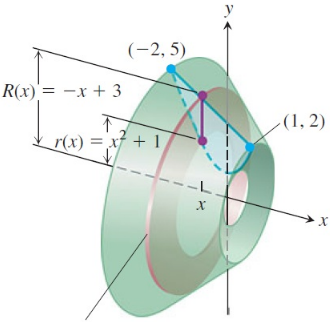
The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Example

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



(a)



Washer cross-section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

Solution We use the four steps for calculating the volume of a solid as discussed early in this section.

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution .

Solution We use the four steps for calculating the volume of a solid as discussed early in this section.

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution .
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.
   These radii are the distances of the ends of the line segment from the axis of revolution .

$$\text{Outer radius:} \qquad R(x) = -x + 3,$$

$$\text{Inner radius:} \qquad r(x) = x^2 + 1.$$

Example 9

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3, \\ x &= -2, \qquad x = 1. \end{aligned}$$

Example 9

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3, \\ x &= -2, \qquad x = 1. \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi \left([R(x)]^2 - [r(x)]^2 \right) \, dx \\ &= \int_{-2}^{1} \pi \left((-x+3)^2 - (x^2+1)^2 \right) \, dx \\ &= \pi \int_{-2}^{1} \left(8 - 6x - x^2 - x^4 \right) \, dx \\ &= \boxed{\frac{117\pi}{5}}. \end{aligned}$$

Practice

- Find the volume of the solid whose base is the disk $x^2 + y^2 \leq 1$ and whose cross-sections perpendicular to the x-axis are squares.

Practice

- Find the volume of the solid whose base is the disk $x^2 + y^2 \leq 1$ and whose cross-sections perpendicular to the x-axis are squares.
- The region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ is revolved about the y-axis. Set up (but do not evaluate) the integral for its volume.

Practice

- Find the volume of the solid whose base is the disk $x^2 + y^2 \leq 1$ and whose cross-sections perpendicular to the x-axis are squares.
- The region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ is revolved about the y-axis. Set up (but do not evaluate) the integral for its volume.
- The region between $y = \sin x$ and $y = 0$ on $[0, \pi]$ is revolved about the x-axis. Find the volume.

Summary

- Slice the solid perpendicular to an axis; if $A(x)$ is the cross-sectional area,
$$\text{Volume} = \int_a^b A(x)\,dx,$$
the same Riemann-sum idea as area under a curve, one dimension up.

Summary

- Slice the solid perpendicular to an axis; if $A(x)$ is the cross-sectional area,
$$\text{Volume} = \int_a^b A(x)\,dx,$$
the same Riemann-sum idea as area under a curve, one dimension up.
- **Cavalieri's Principle:** only $A(x)$ matters, not the shape of the solid — same cross-sectional areas at every level means same volume.

Summary

- Slice the solid perpendicular to an axis; if $A(x)$ is the cross-sectional area,
$$\text{Volume} = \int_a^b A(x)\,dx,$$
the same Riemann-sum idea as area under a curve, one dimension up.
- **Cavalieri's Principle:** only $A(x)$ matters, not the shape of the solid — same cross-sectional areas at every level means same volume.
- For solids of revolution: **disk method** Volume $= \int_a^b \pi R(x)^2\,dx$; **washer method** Volume $= \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx$.