# Lecture 02
# Areas Between Curves

M. Rajesh Kannan and D. Sukumar

Department of Mathematics,
Indian Institute of Technology Hyderabad,
email: rajeshkannan@math.iith.ac.in, suku@math.iith.ac.in

September 2, 2026

# Outline

# Area Under a Curve

- If $f(x) \geq 0$ on $[a, b]$, the area of the region bounded above by $y = f(x)$, below by the $x$-axis, and on the sides by $x = a$, $x = b$ is

$$A = \int_a^b f(x)\, dx.$$

# Area Under a Curve

- If $f(x) \geq 0$ on $[a, b]$, the area of the region bounded above by $y = f(x)$, below by the $x$-axis, and on the sides by $x = a$, $x = b$ is

$$A = \int_a^b f(x)\, dx.$$

- **Today's question:** what if the region lies between *two* curves instead of between a curve and the $x$-axis?

# Area Between Two Curves

## Definition

Let $f$ and $g$ be continuous on $[a, b]$ with $f(x) \geq g(x)$ for all $x \in [a, b]$. The area of the region between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

# Area Between Two Curves

## Definition

Let $f$ and $g$ be continuous on $[a, b]$ with $f(x) \geq g(x)$ for all $x \in [a, b]$. The area of the region between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

- **Riemann sum picture:** slice the region into thin vertical strips of width $\Delta x$. Each strip is (approximately) a rectangle of height $f(c_k) - g(c_k)$ and width $\Delta x_k$, so its area is $\big(f(c_k) - g(c_k)\big)\Delta x_k$.

# Area Between Two Curves

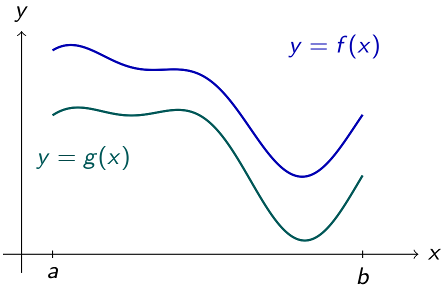
## Definition

Let $f$ and $g$ be continuous on $[a, b]$ with $f(x) \geq g(x)$ for all $x \in [a, b]$. The area of the region between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

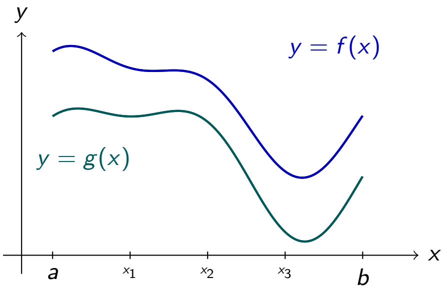
- **Riemann sum picture:** slice the region into thin vertical strips of width $\Delta x$. Each strip is (approximately) a rectangle of height $f(c_k) - g(c_k)$ and width $\Delta x_k$, so its area is $\big(f(c_k) - g(c_k)\big)\Delta x_k$.
- Summing and letting $\|P\| \to 0$ turns the Riemann sum into the integral above. **Upper curve − Lower curve, integrated.**

# Area Between Two Curves: Step by Step



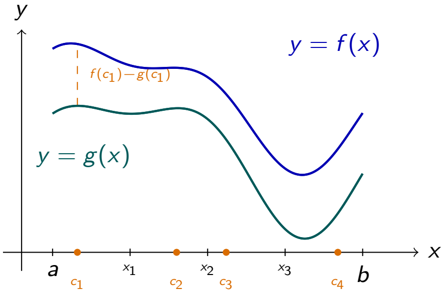

Two curves $f(x) \geq g(x)$ on $[a, b]$.

# Area Between Two Curves: Step by Step



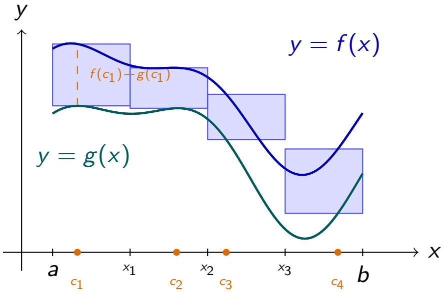

Split $[a, b]$ into subintervals: a partition $P = \{x_0, \ldots, x_4\}$.

# Area Between Two Curves: Step by Step




In each subinterval, choose a point $c_k$.

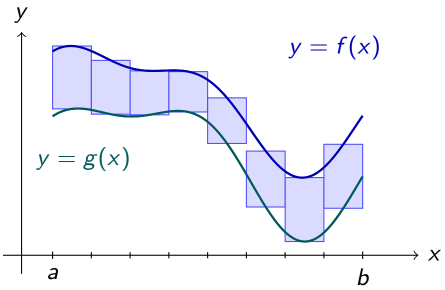
# Area Between Two Curves: Step by Step




Build a strip of height $f(c_k) - g(c_k)$ over each subinterval:

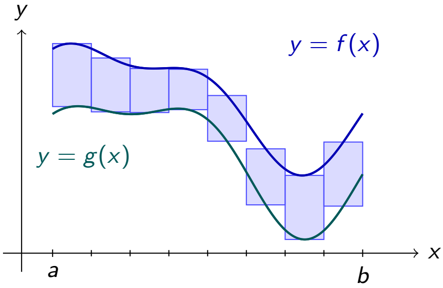
$$S_P = \sum_{k=1}^{4} \big(f(c_k) - g(c_k)\big)\Delta x_k.$$

# Area Between Two Curves: Step by Step



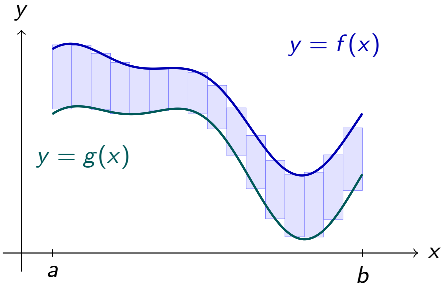

Refine the partition: more, thinner subintervals.

# Area Between Two Curves: Step by Step



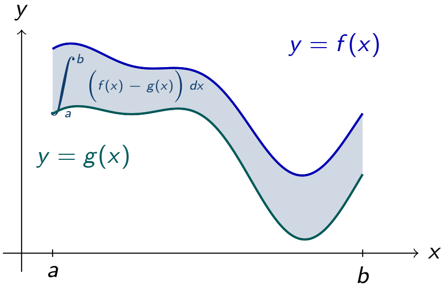

The Riemann sum with the finer partition approximates the area better.

# Area Between Two Curves: Step by Step




Keep refining $P$ . . .

# Area Between Two Curves: Step by Step




As $\|P\| \to 0$, the strips' total area converges to $\displaystyle\int_a^b \big(f(x) - g(x)\big)\,dx$.

# When the Curves Cross

- If $f(x) - g(x)$ changes sign on $[a, b]$ (the curves cross), the formula $A = \int_a^b (f - g)\, dx$ by itself only gives the signed area, not the true geometric area.

# When the Curves Cross

- If $f(x) - g(x)$ changes sign on $[a, b]$ (the curves cross), the formula $A = \int_a^b (f - g)\,dx$ by itself only gives the signed area, not the true geometric area.
- **Recipe:**
  1. Find all points where $f(x) = g(x)$ on $[a, b]$; these split $[a, b]$ into subintervals on each of which one curve stays on top.

# When the Curves Cross

- If $f(x) - g(x)$ changes sign on $[a, b]$ (the curves cross), the formula $A = \int_a^b (f - g)\,dx$ by itself only gives the signed area, not the true geometric area.
- **Recipe:**
  1. Find all points where $f(x) = g(x)$ on $[a, b]$; these split $[a, b]$ into subintervals on each of which one curve stays on top.
  2. On each subinterval, integrate (top $-$ bottom).

## When the Curves Cross

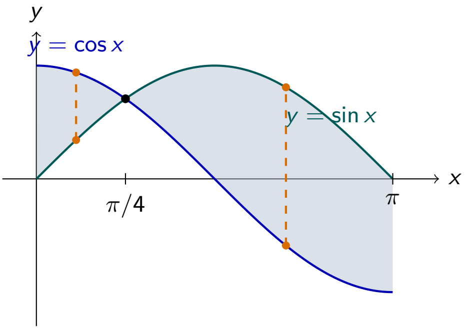
- If $f(x) - g(x)$ changes sign on $[a, b]$ (the curves cross), the formula $A = \int_a^b (f - g)\,dx$ by itself only gives the signed area, not the true geometric area.
- **Recipe:**
  1. Find all points where $f(x) = g(x)$ on $[a, b]$; these split $[a, b]$ into subintervals on each of which one curve stays on top.
  2. On each subinterval, integrate (top − bottom).
  3. Add up the (positive) areas of all the pieces.

## When the Curves Cross

- If $f(x) - g(x)$ changes sign on $[a, b]$ (the curves cross), the formula $A = \int_a^b (f - g)\,dx$ by itself only gives the signed area, not the true geometric area.
- **Recipe:**
  1. Find all points where $f(x) = g(x)$ on $[a, b]$; these split $[a, b]$ into subintervals on each of which one curve stays on top.
  2. On each subinterval, integrate (top − bottom).
  3. Add up the (positive) areas of all the pieces.
- Equivalently, in one line:

$$A = \int_a^b \left|f(x) - g(x)\right| dx.$$

# When the Curves Cross: A Picture




On $[0, \pi/4]$: top $= \cos x$, bottom $= \sin x$.
On $[\pi/4, \pi]$: top $= \sin x$, bottom $= \cos x$.
Each dashed segment is one top$-$bottom strip.

# Integrating with Respect to $y$

- Sometimes it is easier to slice **horizontally**. If the region is bounded on the right by $x = F(y)$ and on the left by $x = G(y)$, for $y \in [c, d]$, with $F(y) \geq G(y)$, then

$$A = \int_c^d \big(F(y) - G(y)\big)\, dy.$$

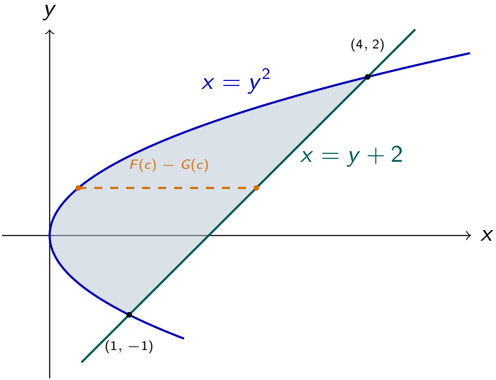
# Integrating with Respect to $y$

- Sometimes it is easier to slice **horizontally**. If the region is bounded on the right by $x = F(y)$ and on the left by $x = G(y)$, for $y \in [c, d]$, with $F(y) \geq G(y)$, then

$$A = \int_c^d \big(F(y) - G(y)\big)\, dy.$$

- **How to decide** $x$ **vs.** $y$**:** sketch the region. If vertical strips would require splitting into several pieces (because the top/bottom boundary changes), but horizontal strips do not (left/right boundary is a single curve throughout), integrate with respect to $y$ instead.

# Integrating with Respect to $y$: A Picture




Left boundary $x = G(y) = y^2$, right boundary $x = F(y) = y + 2$, for $y \in [-1, 2]$. Each dashed segment joins the left curve to the right curve: length $F(y) - G(y)$.

# Worked Example

Find the area of the region enclosed by $y = x^2$ and $y = x + 2$.

# Worked Example

Find the area of the region enclosed by $y = x^2$ and $y = x + 2$.

- **Intersections:** $x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0 \implies x = -1, 2.$

# Worked Example

Find the area of the region enclosed by $y = x^2$ and $y = x + 2$.

- **Intersections:** $x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0 \implies x = -1, 2$.
- **Which curve is on top?** At $x = 0$: line gives 2, parabola gives 0. So $y = x + 2$ is above $y = x^2$ on $[-1, 2]$.

# Worked Example

Find the area of the region enclosed by $y = x^2$ and $y = x + 2$.

- **Intersections:** $x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0 \implies x = -1, 2$.
- **Which curve is on top?** At $x = 0$: line gives 2, parabola gives 0. So $y = x + 2$ is above $y = x^2$ on $[-1, 2]$.
- $$A = \int_{-1}^{2} \left[(x+2) - x^2\right] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}.$$

# Worked Example

Find the area of the region enclosed by $y = x^2$ and $y = x + 2$.

- **Intersections:** $x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0 \implies x = -1, 2.$
- **Which curve is on top?** At $x = 0$: line gives 2, parabola gives 0. So $y = x + 2$ is above $y = x^2$ on $[-1, 2]$.
- $$A = \int_{-1}^{2} \left[(x+2) - x^2\right] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}.$$
- At $x = 2$: $2 + 4 - \frac{8}{3} = \frac{10}{3}$. At $x = -1$: $\frac{1}{2} - 2 + \frac{1}{3} = -\frac{7}{6}$.

## Worked Example

Find the area of the region enclosed by $y = x^2$ and $y = x + 2$.

- **Intersections:** $x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0 \implies x = -1, 2$.
- **Which curve is on top?** At $x = 0$: line gives 2, parabola gives 0. So $y = x + 2$ is above $y = x^2$ on $[-1, 2]$.
- $$A = \int_{-1}^{2} \left[(x+2) - x^2\right] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}.$$
- At $x = 2$: $2 + 4 - \frac{8}{3} = \frac{10}{3}$. At $x = -1$: $\frac{1}{2} - 2 + \frac{1}{3} = -\frac{7}{6}$.
- $$A = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{20}{6} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}.$$

# Practice

- Find the area enclosed by $y = \sin x$ and $y = \cos x$ on $[0, \pi]$.
  **(Curves cross once here — locate the crossing first!)**

# Practice

- Find the area enclosed by $y = \sin x$ and $y = \cos x$ on $[0, \pi]$. **(Curves cross once here — locate the crossing first!)**
- Find the area of the region bounded by $x = y^2$ and $x = y + 2$. **(Try slicing horizontally.)**

# Practice

- Find the area enclosed by $y = \sin x$ and $y = \cos x$ on $[0, \pi]$. **(Curves cross once here — locate the crossing first!)**
- Find the area of the region bounded by $x = y^2$ and $x = y + 2$. **(Try slicing horizontally.)**
- Set up (but do not evaluate) the integral for the area between $y = x^3 - x$ and $y = 0$ on $[-1, 1]$. Where does $f$ change sign?

# Summary

- For $f(x) \geq g(x)$ on $[a, b]$, the area between the curves is

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx,$$

from the Riemann sum of strip areas $\big(f(c_k) - g(c_k)\big)\Delta x_k$.

## Summary

- For $f(x) \geq g(x)$ on $[a, b]$, the area between the curves is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx,$$

  from the Riemann sum of strip areas $\big(f(c_k) - g(c_k)\big)\Delta x_k$.
- If the curves cross, split $[a, b]$ at the intersection points (or use $\int_a^b |f - g|\, dx$) so each piece is genuinely top $-$ bottom.

## Summary

- For $f(x) \geq g(x)$ on $[a, b]$, the area between the curves is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx,$$

from the Riemann sum of strip areas $\big(f(c_k) - g(c_k)\big)\Delta x_k$.
- If the curves cross, split $[a, b]$ at the intersection points (or use $\int_a^b |f - g|\, dx$) so each piece is genuinely top $-$ bottom.
- Sketch the region first: if vertical strips need several pieces but horizontal strips don't, integrate with respect to $y$ instead.