Lecture 01
Riemann Sums and the Definite Integral

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August 31, 2026

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad
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Outline

Riemann Sum

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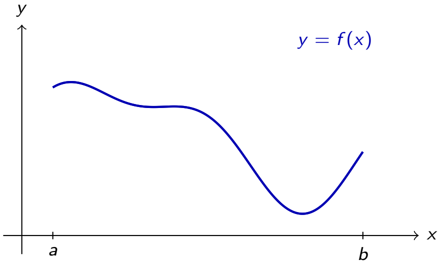
- We can choose the sub-interval lengths to be different too. The **norm** of a partition P, denoted $\|P\|$, is the length of the largest sub-interval.

From a Curve to the Riemann Sum: Step by Step



An arbitrary bounded function f on $[a, b]$.

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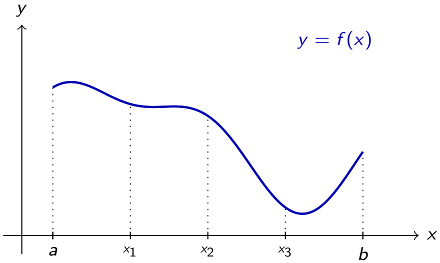

Split $[a, b]$ into subintervals: a partition $P = \{x_0, \dots, x_4\}$.

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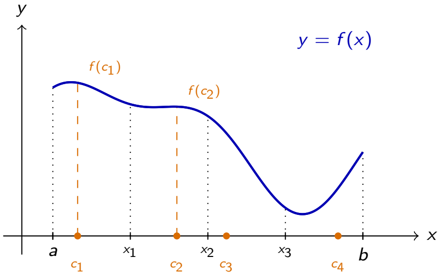

In each subinterval, choose a point c_k.

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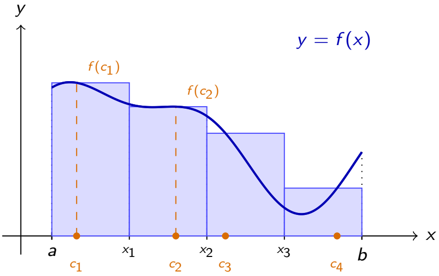

Build a rectangle of height $f(c_k)$ over each subinterval: $S_P = \displaystyle\sum_{k=1}^{4} f(c_k)\Delta x_k$.

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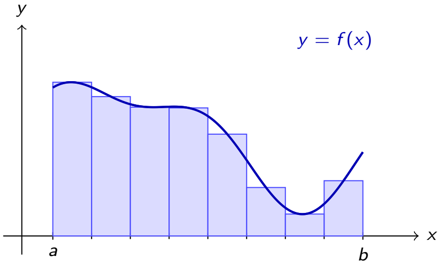
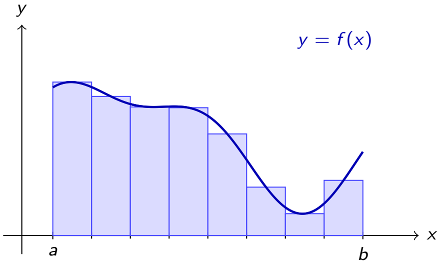


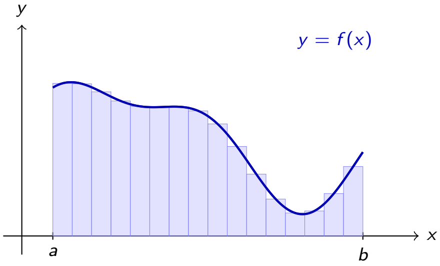
Refine the partition: more, thinner subintervals.

From a Curve to the Riemann Sum: Step by Step



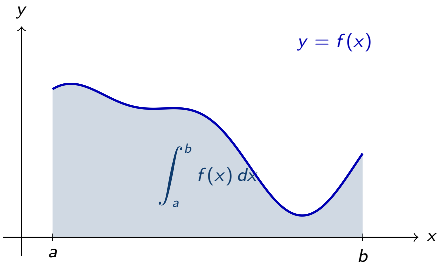
The Riemann sum with the finer partition approximates the area better.

From a Curve to the Riemann Sum: Step by Step



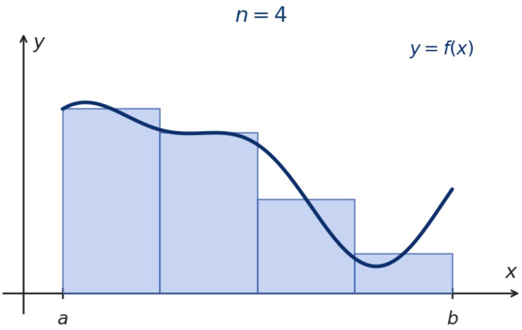
Keep refining P ...

From a Curve to the Riemann Sum: Step by Step



As $\|P\| \to 0$, the rectangles' total area converges to $\displaystyle\int_a^b f(x)\,dx$.

From a Curve to the Riemann Sum: Step by Step



Definition

Consider $f : [a, b] \to \mathbb{R}$. We say f is Riemann integrable, and write $\int_a^b f(x)\,dx = L$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$,

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- If the sub-intervals have equal length,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\frac{b-a}{n}.$$

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- Consider the **Dirichlet function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

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  Then f is **not** Riemann integrable.
- **Why?** For any partition, choosing all c_k rational gives $S_P = b - a$; choosing all c_k irrational gives $S_P = 0$. The Riemann sum does not converge to a single value L.

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Using right endpoints,

$$\begin{aligned}\int_0^1 x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} \\ &= \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} i \\ &= \lim_{n\to\infty} \frac{n(n+1)}{2n^2} \\ &= \boxed{\frac{1}{2}}.\end{aligned}$$

Properties

Theorem

- $\int_a^b f(x)dx = -\int_b^a f(x)dx,$
- $\int_a^a f(x)dx = 0,$
- $\int_a^b (kf(x) + g(x))dx = k\int_a^b f(x)dx + \int_a^b g(x)dx,$
- $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
- *If f has maximum value M and minimum value m on $[a, b]$, then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

- *If f and g are function defined on $[a, b]$ such that $f(x) \leq g(x)$ on $[a, b]$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Proof = Riemann sum!!!

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- Partition $[a, b]$, choose sample points $c_k \in [x_{k-1}, x_k]$, and form the **Riemann sum**

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- Simple functions like $f(x) = 1$, $f(x) = x$, $f(x) = x^2$ are Riemann integrable, but not every bounded function is – the **Dirichlet function** is a standard counterexample.