Basic Iterative methods

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Iterative methods for matrix equations

Let us start with the matrix equation $Ax=b$

i-th row of the matrix represents the equation: $\sum_{j=1}^{n} a_{ij}x_j = b_i$

If we separate out the i-th component: $\sum_{j\neq i} a_{ij}x_j + a_{ii}x_i = b_i$

Now, let us assume a trial solution : $x=x^{(0)}$

And further assume that in each equation, all other terms except the i-th term is evaluated using the trial solution as:

$$\sum_{j\neq i} a_{ij}x_j^{(0)} + a_{ii}x_i = b_i$$

Basic iterative steps

$$\sum_{j\neq i} a_{ij} x_j^{(0)} + a_{ii} x_i = b_i$$

From this, x_i can be evaluated as

$$x_i = \frac{b_i - \sum_{j\neq i} a_{ij} x_j^{(0)}}{a_{ii}}$$

provided $a_{ii} \neq 0$ :: Non-zero diagonals (pivots?)

However, this x_i is not the actual solution as it is calculated using guess values.

We will not get $b_i - \sum_{j=1}^{n} a_{ij} x_j = 0$

Basic iterative steps- Jacobi iteration

So we use an iterative method

$$x_i^{(1)} = \frac{b_i - \sum_{j \neq i} a_{ij} x_j^{(0)}}{a_{ii}}$$

The updated values of x_i are obtained after first iteration

Do this for all x-s, $i=1\ldots n$

Use the updated $x^{(1)}$ as the new guess values and do the next iteration.

$$x_i^{(2)} = \frac{b_i - \sum_{j \neq i} a_{ij} x_j^{(1)}}{a_{ii}}$$

Probably we still end up in $b_i - \sum_{j=1}^{n} a_{ij} x_j \neq 0$

Jacobi iteration

So, we need to update the guess values to be $x^{(3)}$ and carry on the iterations....

$$x_i^{(3)} = \frac{b_i - \sum_{j \neq i} a_{ij} x_j^{(2)}}{a_{ii}}$$

We will still get: $b_i - \sum_{j=1}^{n} a_{ij} x_j^{(3)} \neq 0$

Use the updated x, and carry on iterations for $3,4,5 \ldots$ k-th times

$$x_i^{(k+1)} = \frac{b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}}{a_{ii}}$$

How many steps?----Convergence

After large number of iterations, (k being sufficiently large)

$$b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k)} = 0$$

At this stage the updates solution $x^{(k)}$ converges the to solution $\boldsymbol{x}$ of $Ax=b$.

With further iteration we get: $\max_i \left| x_i^{(k)} - x_i^{(k+1)} \right| < \varepsilon$

ε is a very small number, its value depends on machine precision

Changes in the value of x is infinitesmall at next iterations **-- CONVERGENCE**

Jacobi iterations

Jacobi iteration is an iterative method for matrix solvers

Requirements for successful Jacobi method:

1. Non-zero diagonals, $a_{ii} \neq 0$

2. $$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \qquad \forall i$$

with at least one i for which

3. $$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Iteration step in Jacobi method

Let us look into each row during a Jacobi iteration step

$$x_1^{(k+1)} = \frac{b_1 - \sum_{j\ge 2}^{n} a_{1j} x_j^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21} x_1^{(k)} - \sum_{j\ge 3}^{n} a_{2j} x_j^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31} x_1^{(k)} - a_{32} x_2^{(k)} - \sum_{j\ge 4}^{n} a_{3j} x_j^{(k)}}{a_{33}}$$

$$x_p^{(k+1)} = \frac{b_p - \sum_{j<p}^{n} a_{pj} x_j^{(k)} - \sum_{j\ge p+1}^{n} a_{pj} x_j^{(k)}}{a_{pp}}$$

However, the circled variables are already updated.

We can get faster convergence if the already updated values can be used.

-- Gauss-Seidel iterations

Gauss-Seidel Iterations

A Gauss-Seidel method step uses already updated values as:

$$x_i^{(k+1)} = \frac{b_i - \sum_{j<i}^{n} a_{ij} x_j^{(k+1)} - \sum_{j>i}^{n} a_{ij} x_j^{(k)}}{a_{ii}}$$

$$x_1^{(k+1)} = \frac{b_1 - \sum_{j\ge 2}^{n} a_{1j} x_j^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21} x_1^{(k+1)} - \sum_{j\ge 3}^{n} a_{2j} x_j^{(k)}}{a_{22}}$$

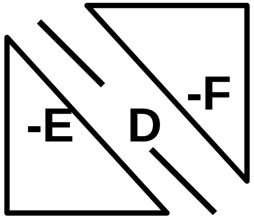
$$x_3^{(k+1)} = \frac{b_3 - a_{31} x_1^{(k+1)} - a_{32} x_2^{(k+1)} - \sum_{j\ge 4}^{n} a_{3j} x_j^{(k)}}{a_{33}}$$

Convergence criterion is same as Jacobi method

Jacobi Iterative methods in matrix form

$Ax=b$

$A=D\text{-}E\text{-}F$



Jacobi step:

$$x_i^{(k+1)} = \frac{b_i - \sum_{j\neq i} a_{ij} x_j^{(k)}}{a_{ii}} = \frac{b_i - \sum_{j<i}^{n} a_{ij} x_j^{(k)} - \sum_{j>i}^{n} a_{ij} x_j^{(k)}}{a_{ii}}$$

$$Dx^{k+1} - Ex^k - Fx^k = b$$

Written in matrix form

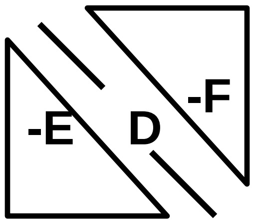
$$\text{or, } Dx^{k+1} = b + \left(E+F\right)x^k$$

$$\Rightarrow x^{k+1} = D^{-1}b + D^{-1}\left(E+F\right)x^k$$

Gauss-Seidel Iterative methods in matrix form

$Ax=b$

$A=D-E-F$



Gauss Seidel step:

$$x_i^{(k+1)} = \frac{b_i - \sum_{j<i}^{n} a_{ij} x_j^{(k+1)} - \sum_{j>i}^{n} a_{ij} x_j^{(k)}}{a_{ii}}$$

$$Dx^{k+1} - Ex^{k+1} - Fx^{k} = b$$

Written in matrix form

$$\text{or, } (D-E)\,x^{k+1} = b + Fx^{k}$$

$$\Rightarrow x^{k+1} = (D-E)^{-1} b + (D-E)^{-1} Fx^{k}$$

General form of an iteration step

$$Mx^{k+1} = Nx^k + b = (M - A)x^k + b$$

For Jacobi: $M=D$, $N=-E-F$

For Gauss-Seidel: $M=D-E$, $N=-F$

General form of an iteration step

$$Mx^{k+1} = (M - A)x^k + b$$

$$\Rightarrow x^{k+1} = M^{-1}(M - A)x^k + M^{-1}b$$

$$\boxed{\Rightarrow x^{k+1} = Gx^k + f}$$

G is called the iteration matrix

If the iteration converges, we get $x=Gx+f$ at the limiting step

$$\Rightarrow x = (I - G)^{-1} f$$

This has a solution if I-G is non-singular!

General form of an iteration step

$$x^{k+1} = Gx^k + f$$

Let x^* be the actual solution

The converged step results in $Ax^*=b \Rightarrow Mx^*=(M-A)x^*+f \Rightarrow x^*=Gx^*+f$

Final step: $$x^* = Gx^* + f$$

$k+1$-th iteration step: $$x^{k+1} = Gx^k + f$$

Subtracting $$x^{k+1} - x^* = G\left(x^k - x^*\right) \qquad (1)$$

iteration steps- convergence requirement

Convergence step: $x^* = Gx^* + f$

k-th iteration step: $x^k = Gx^{k-1} + f$

Subtracting $x^k - x^* = G\left(x^{k-1} - x^*\right)$ (2)

Using (1) and (2) $x^{k+1} - x^* = G\left(x^k - x^*\right) = G^2\left(x^{k-1} - x^*\right)$

Considering all iteration steps

$$x^{k+1} - x^* = G\left(x^k - x^*\right) = G^2\left(x^{k-1} - x^*\right) = G^3\left(x^{k-2} - x^*\right)$$

$$\ldots\ldots = G^{k+1}\left(x^0 - x^*\right)$$

iteration steps- requirement for accurate solution

If x^* is the converged solution and x^0 is an arbitrary initial guess

$$x^{k+1} - x^* = G^{k+1}\left(x^0 - x^*\right)$$

So, $k+1$-th iteration step will converge to $x^{k+1} \sim x^0$ for any x^0 iff:

$$\left\|G^{k+1}\right\| \to 0$$

Or, $$\left\|G\right\| < 1$$

So, if the iteration matrix G has a matrix norm less than 1, the iteration steps will converge to accurate solution of $Ax=b$

iteration steps- convergence Analysis

Let us assume convergence is achieved at $k+1$th step

Convergence step: $x^{k+1} = Gx^{k} + f$

k-th iteration step: $x^{k} = Gx^{k-1} + f$

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1st iteration step: $x^{1} = Gx^{0} + f$

Using the above relations $x^{k+1} - x^{k} = G\left(x^{k} - x^{k-1}\right) = G^{2}\left(x^{k-1} - x^{k-2}\right)$

$$\ldots\ldots = G^{k}\left(x^{1} - x^{0}\right)$$

iteration steps- convergence Analysis

$$x^{k+1} - x^{k} = G\left(x^{k} - x^{k-1}\right) = G^{2}\left(x^{k} - x^{k-2}\right)$$

$$\ldots\ldots = G^{k}\left(x^{1} - x^{0}\right) \qquad \text{(a)}$$

First iteration step $\quad x^{1} = Gx^{0} + f$

Substituting in (a) $\quad x^{k+1} - x^{k} = G^{k}\left(x^{1} - x^{0}\right) = G^{k}\left(Gx^{0} + f - x^{0}\right)$

$$= G^{k}\left(\left(I - G\right)x^{0} + f\right)$$

iteration steps- convergence Analysis

Again, for convergence, x^{k+1} and x^k will have practically same value. I.e., there difference is infinitesmall.

$$\Rightarrow \left| x^{k+1} - x^k \right| < \varepsilon$$

We got $$x^{k+1} - x^k = G^k \left(\left(I - G \right) x^0 + f \right)$$

So, for convergence at k+1-th step: $$\left\| G^{k+1} \right\| \to 0$$

Or, $$\left\| G \right\| < 1$$

Convergence and accuracy of an iterative method

For $Ax=b$, we have the iterative step $x^{k+1} = Gx^k + f$

If $\|G\| < 1$ $\|G^{k+1}\| \to 0$

The iteration will converge for some k as $|x^{k+1} - x^k| < \varepsilon$

The converged solution x^k will be practically same as exact solution x^* as $|x^* - x^k| < \varepsilon$

So, if the iterations converge, they will converge to the exact solution

Diagonally dominant matrices

A matrix A is called

Weakly diagonally dominant if $\left|a_{jj}\right| \geq \sum_{\substack{i=1 \\ i\neq j}}^{n} \left|a_{ij}\right| \qquad \forall j$

Strictly diagonally dominant if $\left|a_{jj}\right| > \sum_{\substack{i=1 \\ i\neq j}}^{n} \left|a_{ij}\right| \qquad \forall j$

Irreducibly diagonally dominant if $\left|a_{jj}\right| \geq \sum_{\substack{i=1 \\ i\neq j}}^{n} \left|a_{ij}\right| \qquad \forall j$

and $\left|a_{jj}\right| > \sum_{\substack{i=1 \\ i\neq j}}^{n} \left|a_{ij}\right|$ at least for one j

Irreducibly / strictly diagonally dominant matrices show non-zero pivots (at least in a permuted form)and hence non-singular

Theorem for convergence of iterations

If A is a strictly diagonally dominant or an irreducibly dominant matrix, then the associated Jacobi or Gauss Seidel iterations converge for any x_0.

$$\begin{bmatrix} 5 & 0 & 4 \\ 1 & 3 & 2 \\ 2 & 6 & 8 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 3 \\ 1 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} 5 & 0 & 4 \\ 1 & 4 & 0 \\ 2 & 5 & 9 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 3 \\ 1 \\ 0 \end{Bmatrix}$$

Can be solved using these methods

Can not be

$$\begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & 2 \\ 2 & 6 & 8 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 3 \\ 1 \\ 0 \end{Bmatrix}$$

Not even The row-permuted form

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 8 \\ 5 & 0 & 4 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 3 \end{Bmatrix}$$

Spectral radius of a matrix

The maximum modulus of the eigenvalues of A is called the spectral radius of A, $\rho(A)$.

For any matrix norm:

$$\lim_{k\to\infty}\left\|A^k\right\|^{1/k} = \rho(A)$$

Convergence for regular splitting of a matrix

Let $A=M-N$.

M,N pair is called a regular splitting if M is non-singular and M^{-1} and N are non-negative.

A non-negative matrix means all elements are non-negative

Theorem: Let M and N be a regular splitting of matrix A. Then $\rho(M^{-1} N)<1$ iff A is non-singular and A^{-1} is non-negative.

The iteration step : $x^{k+1} = M^{-1}Nx^k + M^{-1}b$ will converge if $\rho(M^{-1} N)<1$

This regular splitting with above condition on spectral radius can be only obtained for irreducibly/strictly diagonally dominant matrices ---
from Greshgorin theorem

Theorem on convergence of iterative methods

Let G be a square matrix such that $\rho(G)<1$. Then I-G is non-singular and the iteration $x^{k+1}=Gx^k+f$ converges for every f and x_0.

The converse statement is also true.

Convergence factor

Let the error at k-th step be d_k: $d_k = x^k - x^*$ x^* is the exact solution

$$d_k = G^k d_0$$

Convergence factor (ρ) is given as:

$$\rho = \lim_{k \to \infty} \left(\frac{\|d_k\|}{\|d_0\|} \right)^{1/k}$$

For faster convergence, convergence factor (ρ) must be small

General convergence factor

Convergence factor (ρ) depends on initial guess x_0. General convergence factor, φ, is defined as independent of initial guess.

$$\phi = \lim_{k\to\infty}\left(\max_{x_0\in R^n}\frac{\|d_k\|}{\|d_0\|}\right)^{1/k}$$

$$= \lim_{k\to\infty}\left(\max_{x_0\in R^n}\frac{\|G^k d_0\|}{\|d_0\|}\right)^{1/k}$$

$$= \lim_{k\to\infty}\left(\|G^k\|\right)^{1/k} \qquad = \rho(G) \quad \text{spectral radius}$$

Convergence Rate

For faster convergence, convergence rate (τ) must be high

$$\tau = -\ln(\rho)$$
$$\approx -\ln(\rho(G))$$

Smaller the spectral radius of G, faster the convergence, higher the convergence rate. Less number of iterations will be needed.

At spectral radius≥1, iterations will cease to converge!

Looking into the convergence

Let us run a Gauss-Seidel code for a diagonally dominant (3x3) matrix

We look into the convergence in terms of $L\infty$ norm for residual $r=b\text{-}Ax$ and $x^{(k+1)}\text{-}x^{(k)}$

residual:	1.2400000000000000	x(k+1)-x(k):	0.45000000000000001	ita=	1
residual:	0.42159999999999975	x(k+1)-x(k):	0.24800000000000005	ita=	2
residual:	0.14334400000000014	x(k+1)-x(k):	8.4319999999999951E-002	ita=	3
residual:	4.8736959999999829E-002	x(k+1)-x(k):	2.8668799999999939E-002	ita=	4
residual:	1.6570566400000208E-002	x(k+1)-x(k):	9.7473919999999659E-003	ita=	5
residual:	5.6339925760000575E-003	x(k+1)-x(k):	3.3141132799999751E-003	ita=	6
residual:	1.9155574758400462E-003	x(k+1)-x(k):	1.1267985152000337E-003	ita=	7
residual:	6.5128954178561571E-004	x(k+1)-x(k):	3.8311149516800924E-004	ita=	8
residual:	2.2143844420652314E-004	x(k+1)-x(k):	1.3025790835718976E-004	ita=	9
residual:	7.5289071030271160E-005	x(k+1)-x(k):	4.4287688841349038E-005	ita=	10
residual:	2.5598284150385453E-005	x(k+1)-x(k):	1.5057814206076436E-005	ita=	11
residual:	8.7034166111887856E-006	x(k+1)-x(k):	5.1196568300326817E-006	ita=	12
residual:	2.9591616479418548E-006	x(k+1)-x(k):	1.7406833222599616E-006	ita=	13
residual:	1.0061149600115726E-006	x(k+1)-x(k):	5.9183232969939326E-007	ita=	14
residual:	3.4207908639061202E-007	x(k+1)-x(k):	2.0122299204672345E-007	ita=	15
residual:	1.1630688945274414E-007	x(k+1)-x(k):	6.8415817233713483E-008	ita=	16
residual:	3.9544342023134504E-008	x(k+1)-x(k):	2.3261377934957750E-008	ita=	17
residual:	1.3445076607609963E-008	x(k+1)-x(k):	7.9088683380135194E-009	ita=	18

Convergence history

```
residual:    1.2400000000000000          x(k+1)-x(k):  0.45000000000000001          ita=     1
residual:   0.42159999999999975          x(k+1)-x(k):  0.24800000000000005          ita=     2
residual:   0.14334400000000014          x(k+1)-x(k):   8.4319999999999951E-002     ita=     3
residual:    4.8736959999999829E-002     x(k+1)-x(k):   2.8668799999999939E-002     ita=     4
residual:    1.6570566400000208E-002     x(k+1)-x(k):   9.7473919999999659E-003     ita=     5
residual:    5.6339925760000575E-003     x(k+1)-x(k):   3.3141132799999751E-003     ita=     6
residual:    1.9155574758400462E-003     x(k+1)-x(k):   1.1267985152000337E-003     ita=     7
residual:    6.5128954178561571E-004     x(k+1)-x(k):   3.8311149516800924E-004     ita=     8
residual:    2.2143844420652314E-004     x(k+1)-x(k):   1.3025790835718976E-004     ita=     9
residual:    7.5289071030271160E-005     x(k+1)-x(k):   4.4287688841349038E-005     ita=    10
residual:    2.5598284150385453E-005     x(k+1)-x(k):   1.5057814206076436E-005     ita=    11
residual:    8.7034166111887856E-006     x(k+1)-x(k):   5.1196568300326817E-006     ita=    12
residual:    2.9591616479418548E-006     x(k+1)-x(k):   1.7406833222599616E-006     ita=    13
residual:    1.0061149600115726E-006     x(k+1)-x(k):   5.9183232969939326E-007     ita=    14
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residual:    3.9544342023134504E-008     x(k+1)-x(k):   2.3261377934957750E-008     ita=    17
residual:    1.3445076607609963E-008     x(k+1)-x(k):   7.9088683380135194E-009     ita=    18
```

1. Both the parameters monotonically reduce to zero.
2. The difference of $x^{(k+1)}$-$x^{(k)}$ is correlated with $r=b-Ax$.
3. The changes in $x^{(k+1)}$-$x^{(k)}$ is slows down with increase in iteration number

Numerical experiments

Consider matrix generated as finite difference approximation of

$$\frac{d^2T}{dx^2} = 0; \quad 0 \le x \le 1 \; T(0) = 0, T(1) = 1$$

A tridiagonal matrix is obtained.
Matrix size is varied by varying number of grid points

$$\begin{bmatrix} -2 & 1 & 0 & 0 & . & . & \\ 1 & -2 & 1 & . & & & \\ 0 & 1 & -2 & 1 & & & \\ 0 & 0 & 1 & -2 & 1 & & \\ . & . & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

An example 8x8 matrix

Numerical experiments

Matrices are solved using Jacobi and Gauss Seidel method.

The spectral radius of iteration matrix (G), convergence rate and number of iterations are noted for different sizes of the matrices.

Numerical experiments

Matrix size		Spectral radius of G, $\rho(G)$	Convergence rate, $\eta=-\ln(\rho(G))$	Number of iterations for convergence till $\varepsilon=10^{-6}$
10x10	Jacobi	0.94632	0.05517	200
	Gauss-Seidel	0.89533	0.11034	106
20x20	Jacobi	0.98645	0.01364	709
	Gauss-Seidel	0.97309	0.02728	374
40x40	Jacobi	0.99661	0.00340	2435
	Gauss-Seidel	0.99322	0.00680	1291

Observation:

1. Largest eigenvalue increases with matrix size
2. Spectral radius is higher for Jacobi
3. Number of required iterations are strongly related with convergence rate

How to improve convergence?

Jacobi/ Gauss-Seidel iteration step: $x^{k+1} = Gx^k + f$

$$x^{k+1} - x^k = Gx^k - x^k + f$$

$$= (G - I)x^k + f$$

This is the difference that x covers in a particular iteration till it converges to the exact solution x^*

If $x^{(k+1)}$-$x^{(k)}$ can be increased at each iteration step, number of iterations will be reduced!

Successive over relaxation is proposed

Successive over-relaxation (SOR)

Jacobi/ Gauss-Seidel iteration step: $x^{k+1} - x^k = (G - I)x^k + f$

SOR iteration step: $x^{k+1} - x^k = \omega(G - I)x^k + f \qquad \omega > 1$

At each iteration step, update x as: $x^{k+1} = x^k + \omega(G - I)x^k + f$

SOR can be expressed as $(D - \omega E)x^{k+1} = [\omega F + (I - \omega)D]x^k + \omega b$

$$\Rightarrow x^{k+1} = (D - \omega E)^{-1}[\omega F + (I - \omega)D]x^k + (D - \omega E)^{-1}\omega b$$

So, the iteration matrix: $G = (D - \omega E)^{-1}[\omega F + (I - \omega)D]$

Successive over-relaxation (SOR)- Convergence

For convergence $\rho(G)<1$

Theorem:
If A is symmetric with positive diagonal element and for $0<\omega<2$, SOR converges for any x_0, iff A is positive definite.

If $\omega>2$, SOR will diverge

If $\omega=1$, SOR is same as the basic G-S or Jacobi method.

If $\omega<2$, SOR is actually under-relaxing the iterations or increasing the number of iterations.

Optimum value of ω? When $\rho(G)$ is lease?

Optimum relaxation factor, ω_{opt}

Theorem:

Let A be a consistently ordered matrix such that $a_{ii} \neq 0$, for $i=1,\ldots n$ and let $\omega \neq 0$. Then if λ is a non-zero eigenvalue of the SOR iteration matrix and there is any scalar μ that satisfies

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2$$

Then μ is an eigenvalue of the Jacobi iteration matrix, B.

Conversely, if μ is an eigenvalue of Jacobi iteration matrix, B, and if a scalar λ satisfies $(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2$, then λ is an eigenvalue of the SOR iteration matrix.

Using the above theorem, optimum SOR factor, ω_{opt}, is obtained as:

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B)^2}}$$

Numerical experiments- Gauss Siedel and SOR

	No. of equations (N)	Jacobi	Gauss-Siedel	GS-SOR $\omega=1.2$	GS-SOR $\omega=1.5$	GS-SOR $\omega=1.8$	GS-SOR $\omega=1.9$
No. of iterations to reach a tolerance of 1.0E-06	10	200	106	72	30	81	173
	20	709	374	258	133	83	170
	40	2435	1291	899	479	160	174
Spectral radius $\rho(\mathbf{G})$	10	0.94632	0.89553	0.84206	0.59423	0.80000	0.90000
	20	0.98645	0.97309	0.95956	0.91676	0.80000	0.90000
	40	0.99661	0.99322	0.98983	0.97952	0.92950	0.90000
Convergence rate $\eta=-\ln(\rho(\mathbf{G}))$	10	0.05517	0.11034	0.17190	0.52049	0.22314	0.10536
	20	0.01364	0.02728	0.04128	0.08691	0.22314	0.10536
	40	0.00340	0.00680	0.01022	0.02068	0.07311	0.10536

Obs: No. of iterations are least at a particular SOR factor

Optimum SOR

$$\omega_{opt} = \frac{2}{1+\sqrt{1-(\rho)^2}}$$

Number of Equations (N)	Optimum relaxation parameter for GS-SOR (ω_{opt})
10	1.51145
20	1.71812
40	1.84797