Revision of SVD and Classification of handwritten digits problem

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September 2019

References

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3. **Linear Algebra, A. Ramachandra Rao and P Bhimasankaram, Hindustan book agency.**

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- 16×16 gray scale, as in the above figure,

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- a vector in $\mathbb{R}^{256}$.

Classification by computer of handwritten digits

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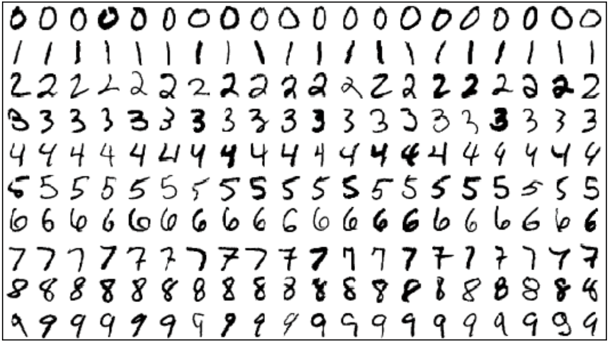

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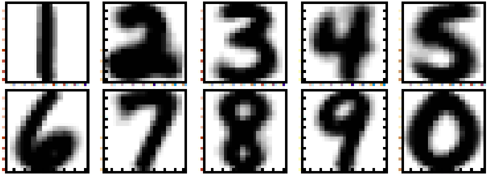
- All the digits of one kind in the training set form a cluster of points in the Euclidean space $\mathbb{R}^{256}$. (Assumption)
- Ideally the clusters are well separated, and the separation between the clusters depends on how well written the training digits are.

The means (“averages“) of all digits in the training set.



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Algorithm

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Using singular value decomposition(SVD), we will see a classification algorithm, for which the success rate is around 93%.

Singular Value Decomposition(SVD)

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$$A = U(\Sigma \ 0)V^T, \text{ where } \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{pmatrix}, \sigma_1 \geq \ldots \geq \sigma_m \geq 0,$$

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and $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal.

The matrix U is called a left singular vector matrix, V is called a right singular vector matrix, and the scalars σ_j are called singular values.

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Fact

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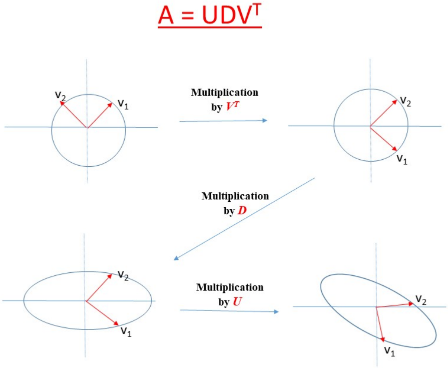
and $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal.

Fact

- *If W is an orthogonal matrix, then $||Wx||_2 = ||x||_2$.*
- *If λ is a non-zero real number, then λ is an eigenvalue of the matrix AA^T if and only if λ is an eigenvalue of A^TA.*

SVD geometry



Computing SVD

Example

Let us compute SVD for the following 2×3 matrix,

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In order to find U, we have to start with AA^T.

$$AA^T = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}.$$

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Next, we have to find the eigenvalues and corresponding eigenvectors of AA^T. After calculating, we get the following eigenvalues and their corresponding eigenvectors.

$$\lambda = 10; \quad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 12; \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

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Thus the matrix A has singular values $\sigma_1 = \sqrt{12}$ and $\sigma_2 = \sqrt{10}$. Now after normalizing u_1 and u_2, we put $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$.

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The calculation of V is similar. V is based on $A^T A$, so we have

$$A^T A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}.$$

Eigenvalues and their corresponding eigenvectors are as follows

$$\textit{for } \lambda = 12; \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

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After normalization, we get $V = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{pmatrix}$ i.e.,

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{pmatrix}$$

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Geometric form of SVD

Let $A \in \mathbb{R}^{m\times n}$ with $m \leq n$. Then, $\mathbb{R}^n$ has an orthonormal basis $\{v_1, \ldots, v_n\}$, $\mathbb{R}^m$ has an orthonormal basis $\{u_1, \ldots, u_m\}$ and there exists $\sigma_1 \geq \sigma_2 \geq \ldots, \geq \sigma_r \geq 0$ such that

$$Av_i = \begin{cases} \sigma_i u_i, & \text{if } i = 1, \ldots, r, \\ 0 & \text{if } i \geq r+1, \end{cases}$$

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and

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Four fundamental subspaces

For an $m \times n$ matrix A, the following subspaces are called fundamental subspaces.

- ***Range space of** A**:*** $R(A) = \{x \in \mathbb{R}^m : x = Ay \text{ for some } y \in \mathbb{R}^n\}$. (span of columns of A)

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Basis for fundamental subspaces

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Illustration

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} =$$

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- $R(A) = span \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\},$

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SVD - equivalent (and useful) form

Theorem

Let $A \in \mathbb{R}^{m \times n}$, and let $\sigma_1, \dots, \sigma_r$ be the nonzero singular values of A, with associated right and left singular vectors $v_1, \dots, v_r$ and $u_1, \dots, u_r$, respectively. Then

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T.$$

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If $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}$, then

$$A = (\sqrt{12}) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{30}} \right) + (\sqrt{10}) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{30}} \right).$$

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Properties

Fact (Magnification)

If $A \in \mathbb{R}^{m \times n}$ has singular values $\sigma_1 \geq \ldots \geq \sigma_p$, where $p = min\{m, n\}$, then

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Algorithm

Training: For the training set of known digits, compute the SVD of each set of digits of one kind.
Classification: For a given test digit, compute its relative residual in all 10 bases. If one residual is significantly smaller than all the others, classify as that. Otherwise give up.

Thank you!

Thank you! Happy journey!!!