Linear Algebra

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References

A. Ramachandra Rao and P Bhimasankaram , Linear Algebra, Second edition, Hindustan book agency.

Outline

- Eigenvalues and eigenvectors,

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- Spectral representation of semi-simple matrices,

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- Singular value decomposition.

Eigenvalues and eigenvectors

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A complex number λ is said to be an eigenvalue of an $n \times n$ complex matrix A, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.

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A complex number λ is an eigenvalue of a complex matrix A if and only if λ is a root of the characteristic polynomial $\det(A - \lambda I) = 0$.

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Definition

Two $n \times n$ matrices A and B are said to be similar, if there exists an invertible matrix C such that $B = C^{-1}AC$.

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Geometric multiplicity of an eigenvalue λ of a matrix A is defined as the dimension of the eigenspace associated with λ. An eigenvalue λ is said to be regular, if its algebraic multiplicity is equal the geometric multiplicity.

A.M. $\geq$ G.M.

Theorem

For any eigenvalue λ of A, the algebraic multiplicity of λ with respect to A is greater than or equal to the geometric multiplicity of λ, as an eigenvalue of A.

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Proof

- Let $\{x_1, \ldots, x_k\}$ be a basis for the eigenspace of λ, and let $\{x_1, x_2, \ldots, x_n\}$ be an extension to a basis of $\mathbb{C}^n$.

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- Set $P = [x_1 \; x_2 \; \ldots \; x_n]$.

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- Set $P = [x_1 \ x_2 \ \ldots \ x_n]$.
- Then P is nonsingular, and

$$\begin{aligned} P^{-1}AP &= P^{-1}[Ax_1 \ Ax_2 \ \ldots \ Ax_k \ \ldots \ Ax_n] \\ &= P^{-1}[\lambda x_1 \ \lambda x_2 \ \ldots \ \lambda x_k \ \ldots \ Ax_n]. \end{aligned}$$

Proof cont...

- Thus,

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- Hence, $\chi_A(\alpha) = \chi_{P^{-1}AP}(\alpha) = (\lambda - \alpha)^k \chi_C(\alpha)$.
- Thus, the algebraic multiplicity of λ is greater than or equal geometric multiplicity of λ.

Properties

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- If λ is an eigenvalue of A with eigenvector x, then λ^k is an eigenvalue of A^k with eigenvector x.
- If $f(\alpha)$ is a polynomial and λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.
- If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A, and $x_1, x_2, \ldots, x_k$ are the corresponding eigenvectors. Then the $x_1, x_2, \ldots, x_k$ are linearly independent.

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- If λ is a nonzero eigenvalue of a square matrix AB(A and B need not be square), then λ is an eigenvalue of the matrix BA with the same algebraic and geometric multiplicities.

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- If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A, and $x_1, x_2, \ldots, x_k$ are the corresponding eigenvectors. Then the $x_1, x_2, \ldots, x_k$ are linearly independent.
- If λ is a nonzero eigenvalue of a square matrix AB(A and B need not be square), then λ is an eigenvalue of the matrix BA with the same algebraic and geometric multiplicities. If $x_1, x_2, \ldots, x_r$ are linearly independent eigenvectors of AB corresponding to to λ, then $Bx_1, \ldots, Bx_r$ are linearly independent eigenvectors of BA corresponding to λ.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

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- Assume the result is true for $(n-1) \times (n-1)$ matrices.

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Proof

- Proof by induction. If $n = 1$, then we are done.
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- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.

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- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.
- Let P be a nonsingular matrix with x as the first column.
- Then, $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some $1 \times n-1$ vector y^T and $(n-1) \times (n-1)$ matrix C.

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- By induction, there exists a non-singular matrix W such that $T = W^{-1}CW$ is upper triangular.

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- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.
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- Set $Q = \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix}$.

Proof cont...

- $(PQ)^{-1}A(PQ) = \begin{pmatrix} 1 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} =$
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Corollary

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the eigenvalues of the matrix* A *and let* $f(\alpha)$ *be a polynomial. Then* $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_k)$ *are the eigenvalues of* $f(A)$*.*

Corollary

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of the matrix A and let $f(\alpha)$ be a polynomial. Then $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_k)$ are the eigenvalues of $f(A)$.

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Proof Exercise!

Cayely-Hamilton Theorem

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- By weak Schur's lemma, $T = PAP^{-1}$ for some invertible matrix P and upper triangular matrix T.

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- Then first k columns of W_k are zero.

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- $k = 1$ trivial.
- Assume the result is true for $k - 1$.

□

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- Then, for $l \leq k$, we have $(W_k)_{il} = \sum_{j=1}^{n} b_{ij}c_{jl} = 0$.(Reason: $b_{ij} = 0$ if $j \leq k-1$ and $c_{jl} = 0$ if $j \geq k$)

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- Thus first k columns of W_k are zero, and hence $W_n = 0$.
- $0 = f(T) = P^{-1}f(A)P$.

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The minimal polynomial of A divides every polynomial which annihilates A.

Proof.

Division algorithm. □

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- Let $\mu(z)$ be the minimal polynomial of A.

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- Let $\mu(z)$ be the minimal polynomial of A.
- If λ is a characteristic root of A, then $\mu(A)x = \mu(\lambda)x$, where x is the eigenvector corresponding to the eigenvalue λ of A.
- As $\mu(A) = 0$, so $\mu(\lambda) = 0$.

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Observation

If $P^{-1}AP = D$ is a diagonal matrix, then $AP = DP$. We can see that, d_i is an eigenvalue of A with i^{th} column of P as the corresponding eigenvector. Conversely, if A has n linear independent eigenvectors, and P is the matrix formed with these vectors as eigenvectors, then $P^{-1}AP$ is diagonal.

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- (2) implies (3) is clear.

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Proof

- (1) implies (2) is clear.
- (2) implies (3) is clear.
- (2) implies (1) follows from the observation.

Applications:

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2. Any idempotent matrix is diagonalizable($P^2 = P$).
3. Any nonzero nilpotent matrix is not diagonalizable($A^k = 0$, for some integer k).

Spectral representation for semi-simple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple and has rank r,*
2. *there exists a non-singular matrix P of order n, and a diagonal nonsingular matrix Δ of order r such that* $A = P \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$,
3. *There exists nonzero scalars $\gamma_1, \gamma_2, \ldots, \gamma_r$ and vectors $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ in $\mathbb{C}^n$ such that ${v_i}^T u_j = \delta_{ij}$ for all i, j and*

$$A = \sum_{i=1}^{r} \gamma_i u_i {v_i}^T$$

Proof

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- (2) implies (3). Set $\delta_i = i^{th}$ diagonal entry of Δ, $u_i = i^{th}$ column of P, and ${v_i}^T = i^{th}$ row of P^{-1} .

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- (3) implies (2). Exercise!

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An $n \times n$ *matrix is said to be symmetric, if* $A^T = A$. *An* $n \times n$ *matrix is said to be Hermitian, if* $A^* = A$.

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An $n \times n$ matrix is said to be symmetric, if $A^T = A$. An $n \times n$ matrix is said to be Hermitian, if $A^* = A$.

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A real matrix A is said to be orthogonal if $AA^T = A^T A = I$, and a complex matrix A is said to be unitary if $AA^* = A^*A = I$.

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- Let λ be a real eigenvalue of A and x be a corresponding eigenvector with unit length.
- Let P be an orthogonal matrix with x as the first column.
- Then $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some vector y^T and some $(n-1) \times (n-1)$ matrix C.

Proof continued...

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- Set $Q = diag(1, W)$, then Q and PQ are diagonal matrices.
- $(PQ)^{-1}A(PQ) = \begin{pmatrix} 1 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & D \end{pmatrix}$
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□

Spectral theorem for Hermitian matrices and Schur's lemma

Theorem

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Similar to spectral theorem real symmetric matrices. □

Spectral decomposition

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Let A be an $n \times n$ Hermitian matrix with rank r. Then A can be represented in each of the following equivalent forms:

1. *There exists a unitary matrix P and a real diagonal nonsingular matrix Δ of rank r such that* $A = P \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} P^*$.

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3. *There exists matrices R and Δ of orders $n \times r$ and $r \times r$, respectively, such that Δ is real, diagonal and non-singular, $R^* R = I$ and $A = R \Delta R^*$.*

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1. *True or False: Every real matrix is orthogonally similar to an upper triangular matrix. Answer : False*
2. *Every real matrix A with real eigenvalues is orthogonally similar to an upper triangular matrix.*

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- By a similar argument, we can conclude T must be diagonal.

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- Now, $T = U^*AU$, and T is normal.
- So, T is diagonal, by previous theorem.

Positive Semidefinite Matrices(PSD)

Let $\mathcal{S}^n$ denote the subspace of symmetric matrices in $\mathbb{R}^{n\times n}$. $A \in \mathcal{S}^n$ is *positive semidefinite*(PSD) if $x^T Ax \geq 0$ for every $x \in \mathbb{R}^n$.

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- (4) implies (1).

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A singular value decomposition of an $m \times n$ *matrix* A *is a representation of* A *in the following form:* $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$*, where* U *and* V *are unitary matrices and* Δ *is a diagonal matrix with positive diagonal entries.*

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- Define $S = BG$ where $G = diag(\frac{1}{\sqrt{d_1}}, \ldots, \frac{1}{\sqrt{d_r}})$.
- Verify $RG^{-1}S^*$ is a singular value decomposition for A.