Annual Foundation School - I
AFS-I@ IIT Hyderabad
Topology

Date: December 9, 2024 Tutorial 1 Week 2

Notation used in the paper:

1.	$\mathbb{R}_u$	the space of real numbers with the usual topology
2.	$\mathbb{R}_i$	the space of real numbers with the indiscrete topology
3.	$\mathbb{R}_d$	the space of real numbers with the discrete topology
4.	$\mathbb{R}_{cc}$	the space of real numbers with the cocountable topology
5.	$\mathbb{R}_{cf}$	the space of real numbers with the cofinite topology
6.	$\mathbb{R}_l$	the space of real numbers with the lower limit topology
7.	$\mathbb{R}_K$	the space of real numbers with K-topology

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1. Let X be an uncountable set. Let

$$\mathcal{T}_c = \{\emptyset\} \cup \{U \subset X \mid X - U \text{ is countable}\}.$$

   (a) Show that $\mathcal{T}$ is a topology on X. This topology is called the *co-countable topology*.

   (b) Show that the finite intersection of non-empty open sets is non-empty. Is the result true in usual topology on $\mathbb{R}$?

2. Let X be a topological space. Let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

3. Let X be a set. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

   a topology on X?

4. Show that if $\{\mathcal{T}_\alpha\}$ is the family of topologies on X, show that $\cap\mathcal{T}_\alpha$ is a topology on X. Is $\cup\mathcal{T}_\alpha$ a topology on X?

5. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

6. If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ and } \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

7. Let $\mathbb{N}$ be the set of natural number. Let $A_n = \{m \in \mathbb{N} \mid m \geq n\}$. Let $\mathcal{T}_1 = \{\emptyset\} \cup \{A_n\}_{n\in\mathbb{N}}$. Let $\mathcal{T}_f$ be the co-finite topology on $\mathbb{N}$. Show that $\mathcal{T}_1$ is a topology on $\mathbb{N}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_f$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_f$.

8. Let $\mathbb{Z}$ be the set of integers. Let $a, b \in \mathbb{Z}$, $b \neq 0$. Denote

$$N_{a,b} = \{a + bn \,|\, n \in \mathbb{Z}\}.$$

A non-empty subset U of $\mathbb{Z}$ we define to be open if it is a union of sets of the form $N_{a,b}$ or U is the empty set. Let $\mathcal{T}$ denotes this collection on subsets of $\mathbb{Z}$.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

(b) Show that $N_{a,b}$ are both open and closed in $\mathcal{T}$.

(c) Is $\{1, -1\}$ open in $\mathcal{T}$?

(d) Using this show that there are infinitely many prime.

9. Let $A_1 = \{\frac{1}{n} \,|\, n \in \mathbb{N}_{>0}\}$, $A_2 = \mathbb{Q}$, $A_3 = [0, 1)$ and $A_5 = (0, 1]$ subsets of $\mathbb{R}$. For any set A we denote the interior of A by A^o and the closure of A by $\bar{A}$. Complete the following table:

Topology on $\mathbb{R}$	A_1^o	$\bar{A}_1$	A_2^o	$\bar{A}_2$	A_3^o	$\bar{A}_3$	A_4^o	$\bar{A}_4$
$\mathbb{R}_u$								
$\mathbb{R}_i$								
$\mathbb{R}_d$								
$\mathbb{R}_{cc}$								
$\mathbb{R}_{cf}$								
$\mathbb{R}_l$								
$\mathbb{R}_K$								

10. Let $(\frac{1}{n})_{n\geq 1}$, $\{(-1)^n\}_{n\geq 1}$ be sequences in $\mathbb{R}$. To what point (s) does these sequences converge to (if there is any) with respect to:

(a) the indiscrete topology

(b) the discrete topology

(c) the cocountable topology

(d) the cofinite topology

(e) the lower limit topology

(f) the K-topology

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Date: December 10, 2024 | Tutorial 2 | Week 2

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1. Let X be an infinite set and τ_{cc}, τ_{cf} be the *co-countable* and the *co-finite* topology on X respectively. Prove that $\tau_{cf} \subset \tau_{cc}$. Moreover, show that the identity map from (X, τ_{cc}) to (X, τ_{cf}) is continuous.

2. Consider $\mathbb{R}^2$ with discrete topology. What is the closure of open unit disc in this topology?

3. Let τ_1 and τ_2 be two topologies on a set X such that $\tau_1 \subset \tau_2$. Prove that
(a)$\overline{A}^{\tau_2} \subset \overline{A}^{\tau_1}$
(b)$\text{int}A^{\tau_1} \subset \text{int}A^{\tau_2}$
for any subset A of X. Here $\overline{A}^{\tau_1}, \overline{A}^{\tau_2}$ denote the closure of A with respect to τ_1 and τ_2 respectively.

4. Show that the identity map $\text{Id} : (X, \tau_d) \to (X, \tau)$ is continuous, where τ_d denotes the discrete topology.

5. Let X be a set and let τ_1 be a topology on X which is different from the discrete topology on X. Then the identity map $\text{Id} : (X, \tau_1) \to (X, \tau_d)$ is not continuous.

6. Define a relation on the set of all topological spaces: for two topological spaces X and Y, $X \sim Y$ if there exists a homeomorphism from X to Y. Show that $\sim$ is an equivalence relation.

7. Prove that the identity map $\text{Id} : \mathbb{R}_\ell \to \mathbb{R}$ is continuous, but not a homeomorphism. Here $\mathbb{R}$ denotes the the usual topology on the real line.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that f as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$ is continuous.

9. Let us define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+2 & \textit{if } x < -2, \\ -x-2 & \textit{if } -2 \leq x < 0, \\ x+2 & \textit{if } x \geq 0. \end{cases}$$

   (a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is not continuous.
   (b) Is the above $f : \mathbb{R}_\ell \to \mathbb{R}$ continuous?
   (c) Discuss continuity of $f : \mathbb{R}_\ell \to \mathbb{R}_\ell$ continuous?

10. Prove that the exponential map $\exp : [0, 2\pi) \to \mathbb{S}^1$ is not open.

11. Show that $[0, 1)$ is open in $([0, 2], \tau)$ where τ is the usual topology on $[0, 2]$.

12. Show that the order topology on $\mathbb{Q}$ is same as subspace topology on $\mathbb{Q}$.

13. Are $[0, 1)$ and $(0, 1]$ homeomorphic? Find an explicit homeomorphism from $[0, 1)$ to $(0, 1]$.

14. (a) Show that $[a, b]$ and $[c, d]$ are homeomorphic.

    (b) Find an explicit homeomorphism from $[a, b)$ to $(c, d]$.

15. (a) Prove that $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.

    (b) Prove that $SL_n(\mathbb{R})$ is a closed subset of $M_n(\mathbb{R})$.

16. Let $f : X \to Y$ be map between topological spaces. Show that f is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y $\Leftrightarrow$ $f(\overline{A}) \subset \overline{f(A)}$, for every subset A of X.

17. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$.

18. A subset A of a topological space X is said to be nowhere dense if $\overline{A}$ has empty interior.

    (a) Show that A is nowhere dense $\Leftrightarrow$ every non-empty open set has a non-empty open subset disjoint from A.

    (b) Show that a closed set is nowhere dense $\Leftrightarrow$ its complement is everywhere dense. Is this true for an arbitrary set?

    (c) how that boundary of a closed set is nowhere dense. Is this true for an arbitrary set?

19. Prove that $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$.

20. Show that set of all traceless 2×2 matrices is nowhere dense.

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1. Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} X_n$$

where $X_n = \mathbb{R}$ for each n. Let us define a function $f : \mathbb{R} \to \mathbb{R}^\omega$ by the equation

$$f(t) = (t, t, t, \dots)$$

the n^{th} coordinate function of f is the function $f_n(t) = t$. Prove that the function f is continuous if $\mathbb{R}^\omega$ is given the product topology, but f is not continuous if $\mathbb{R}^\omega$ is given the box topology. Is it continuous if $\mathbb{R}^\omega$ is given the uniform topology?

2. Consider the product, uniform and box topologies on $\mathbb{R}^\omega$. In which topologies are the following function form $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$g(t) = (t, 2t, 3t, \dots),$$
$$h(t) = (t, \tfrac{t}{2}, \tfrac{t}{3}, \dots).$$

3. Consider the following sequence $\{w_n\}$ in $\mathbb{R}^\omega$.

$$w_1 = (1, 1, 1, 1, \dots)$$
$$w_2 = (0, 2, 2, 2, \dots)$$
$$w_3 = (0, 0, 3, 3, \dots)$$
$$\vdots$$

Check the convergence of the sequence under product, uniform and box topology on $\mathbb{R}^\omega$.

4. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots .$$

(a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.
(b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.
(c) Show that

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

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5. Let $f, g : [0,1] \to X$ be two continuous maps into a topological space X. Assume that $f(1) = g(0)$. Prove that the function $h : [0,1] \to X$ defined by

$$h(t) = \begin{cases} f(2t) & \textit{if } 0 \leq t \leq 1/2, \\ g(2t-1) & \textit{if } 1/2 \leq t \leq 1, \end{cases}$$

is continuous. h is called *concatenation of two paths* f and g.

6. Let $f, g : X \to \mathbb{R}$ be two bounded continuous functions. Prove that $\max(f, g)$ is continuous.

7. Let X and Y be two topological spaces. Give a countable family $\{F_n \mid n \in \mathbb{N}\}$ of closed sets of X such that $f|_{F_n} : F_n \to Y$ is continuous for all n, but $f : \cup_{n \in \mathbb{N}} F_n \to Y$ is not continuous. Is it contradicting Pasting Lemma?

8. (a) Let A be a subspace of a topological space X. Then A is said to be a *retract* of X, if there exists a continuous map $r : X \to A$ such that $r|_A = \text{Id}$. The map r is called a *retraction*. Show that every retraction is a quotient map.
   (b) Prove that the *radial projection* $r : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$ defined by

   $$r(x) = x/||x||$$

   is a quotient map.
   (c) Prove that open maps and closed maps are quotient maps.

9. Give an example of a quotient map
   (a) Which is open but not closed.
   (b) Which is closed but not open.
   (c) which is neither open nor closed.

10. Show that the map $\pi : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ defined by

$$\pi(x, y) = (e^{ix}, e^{iy})$$

is a quotient map.

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Date: December 16, 2024 | Tutorial 1 | Week 3

Notation used in the paper:

1.	τ	be the usual topology	2.	τ_{di}	be discrete topology
3.	τ_{cc}	be the cocountable topology	4.	τ_{cf}	be the cofinite topology

1. Discuss the compactness of the below sets
   (a) $((0,1),\tau)$
   (b) $(\mathbb{D}^n,\tau)$
   (c) $(\mathbb{Q},\tau)$
   (d) $A=\{\frac{1}{n}, n\in\mathbb{N}\}$.
   (e) $A=\{\frac{1}{n}, n\in\mathbb{N}\}\cup\{0\}$.
   (f) $(\mathbb{R},\tau_{cf})$.
   (g) $(\mathbb{R},\tau_{cc})$.
   (h) $[0,1],\tau_{cc})$.
   (i) (X,τ_{cc}), X is countable.
   (j) Under what condition (X,τ_{cc}) is compact?
   (k) $([0,1],\tau_k)$
2. Prove or disprove the statement below.
   (a) (X,τ_{di}) is compact iff X is finite.
   (b) The finite union of compact sets is compact.
   (c) The arbitrary union of compact sets is compact.
   (d) The intersection of compact sets in Hausdroff space is compact.
   (e) What happens if we remove the Hausdroff condition?
3. Let X, Y are topological spaces. If $X\times Y$ is compact. Then X and Y are also compact. What about the converse of the above statement?
4. Let $f:(\mathbb{R},\tau_{cf})\to(\mathbb{R},\tau_{di})$, defined by $f(x)=x^2$. Then
   (a) Is $f(\mathbb{R})$ compact
   (b) What can you infer from the above statement?
5. Which of the following topological space is locally compact.
   (a) $\mathbb{R}$ with usual topology.
   (b) $\mathbb{Q}$ with usual topology.
   (c) X with discrete topology.
6. Let (X,τ) be a compact metric space, then prove that (X,τ) is separable. Also prove that it is second countable.
7. Give an example of topological space which is limit point compact but not compact.
8. Is the union of two locally compact spaces locally compact. If not, give a counterexample to show that.

9. Are polynomials functions, exponential functions, and sin functions proper functions?

10. The function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = xy$ is not proper.

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Date: December 16, 2024 Tutorial 2 Week 3

Notation used in the paper:

1.	τ	be the usual topology	2.	τ_{di}	be discrete topology
3.	τ_{cc}	be the cocountable topology	4.	τ_{cf}	be the cofinite topology
5.	τ_k	be the k-topology.	6.	$\mathbb{R}_l$	be the topological space $\mathbb{R}$ with lower limit topology.

1. Let $f, g : X \to Y$ be continuous functions between two topological spaces. Consider the set $F := \{ x \in X | f(x) = g(x) \}$. If Y is a Hausdorff space, then F must be closed in X.
2. If $\Pi_{\alpha \in \Lambda} X_\alpha$ (non-empty) is Hausdorff, then every X_α is Hausdorff for every $\alpha \in \Lambda$.
3. Limit of a sequence in a Hausdorff space is unique. Is the converse true?
4. Consider a compact topological group G acting on X, a topological space. Let X/G denote the set of orbits of the given actions
   (a) If X is Hausdorff, then prove that X/G is Hausdorff.
   (b) Prove that the same is true for regular and normal spaces.
5. Let X be a locally compact Hausdorff space and $A \subseteq X$ be a subspace. Prove that if $A \subseteq X$ is closed or open, then A is locally compact.
6. Consider a proper surjective map $f : X \to Y$ function. If X is Hausdorff, then show that Y is also Hausdorff.
7. Prove $\mathbb{R}_l$ is normal.
8. Prove that the metric spaces are normal.
9. If X is a Hausdroff space and A is a compact subset of X. Then A is closed.
10. Consider $f : X \to Y$ is a continuous function. If X is compact and Y is Hausdorff, then f is proper.
11. Two compact subsets of a Hausdorff space can be separated by open sets.
12. A topological space is said to be **perfectly normal** if for any closed subset $A \subseteq X$, there exists a continuous function $f : X \to \mathbf{R}$ such that $f^{-1}(0) = A$. Then prove that
    (a) Every metric space is perfectly normal
    (b) X perfectly normal $\implies$ X is normal.
13. Give a proof of the Urysohn lemma for metric spaces.

14. Let X be completely regular and A, B be disjoint closed subsets of X. Show that if A is compact, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

15. If X is a connected normal space with at least two elements, then X is uncountable.

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