INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
Annual Foundation School - 1
Topology 2 -28, December 2024

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1. Let X be a metric space with metric d. Show that d_1, defined by

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

   is also a metric on X. Observe that X itself is a bounded set in the metric space (X, d_1).

2. Let $X = \{0,1\}^n$ (the boolean cube), the set of all strings of length n with entries are from $\{0,1\}$. For $x, y \in X$, define $d(x,y)$ to the number of coordinates in which x and y differ. Show that d is a metric.

3. Let G be a group. Let $\{a, b, a^{-1}, b^{-1}\}$ be a generating set for G. Define $d(v,w)$ is the minimal k such that k such that $v = wg_1g_2\cdots g_k$, where $g_i \in \{a, b, a^{-1}, b^{-1}\}$ for all i. Show that d is a metric on G.

4. Let G be a finite graph. Suppose G is connected. Define $d(v,v) = 0$ and $d(v,w)$ to be the length of the shortest path between the vertices v and w. Show that d is a metric.

5. Consider $\mathbb{Z}$, the set of all integers. Define $d(x,y)$ to be 2^{-m} for $x \neq y$, where m is the largest power of two dividing $x-y$ and $d(x,x) = 0$. Show that d is a metric. Calculate $d(999, 1000)$ and $d(0, 1000)$. Show also that $d(x,z) \leq \max\{d(x,y), d(y,z)\}$.

6. Let (X, d_X) and (Y, d_Y) be metric spaces. Then, show that $X \times Y$ is a metric space with respect the following: $d((x_1,y_1),(x_2,y_2)) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$.

7. Consider a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 0 & \textit{if } x = y, \\ |x| + |y| & \textit{if } x \neq y. \end{cases} \tag{1}$$

   Show that d is a metric on $\mathbb{R}$.

8. Let X be a non-empty set and let d be a real function of ordered pairs of elements of X which satisfies the following two conditions: $d(x,y) = 0 \Longleftrightarrow x = y$ and

$$d(x,y) \leq d(x,z) + d(y,z)$$

   for all $x, y, z \in X$. Show that d is a metric on X.

9. Show that every Cauchy sequence is bounded.

10. What are the Cauchy sequences in a discrete metric space? Show that every discrete metric space is complete.

11. Define d on $\mathbb{N}$ as follows: $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$. Show that d defines a metric on $\mathbb{N}$. Show that this is not a complete metric space.

12. Let (X, d) be a metric space and $\mathcal{B}(X, \mathbb{R})$ be the set of all bounded real functions defined on X. Prove that it is a metric space with respect to the metric induced by the norm
$$||f|| = \sup_{x \in X} |f(x)|.$$
Show that this metric space is complete.

13. If A and B are open subset of $\mathbb{R}^n$, then $A + B$ is also open where
$$A + B = \{x + y | x \in A, y \in B\}.$$

14. Describe the interior of each of the following subsets of the real line: the set of all integers; the set of all rationals; the set of all irrationals; $(0,1)$; $[0,1]$; $[0,1) \cup \{1,2\}$. Do the same for each of the following subsets of the complex plane:
$$\{z : |z| < 1\}; \{z : |z| \leq 1\}; \{z : I(z) = 0\}; \{z : Re(z) \text{ is rational}\}.$$

15. Let A and B be two subsets of a metric space X and prove the following:
(a) $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$;
(b) $\text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B)$.

    Give an example of two subsets A and B of the real line such that
$$\text{Int}(A) \cup \text{Int}(B) \neq \text{Int}(A \cup B).$$

16. Let X be a metric space, and let G be an open set in X. Prove that G is disjoint from a set $A \Leftrightarrow G$ is disjoint from $\bar{A}$.

17. Show that the following functions f_n defined on $[0,1]$ is a Cauchy sequence in the space $(C(X, \mathbb{R}), d)$, (continuous bounded functions on X) where $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$:
$$f_n(x) = \begin{cases} 1 & \textit{if } 0 \leq x \leq 1/2, \\ -2^n(x - 1/2) + 1 & \textit{if } 1/2 \leq x \leq 1/2 + (1/2)^n, \\ 0 & \textit{if } 1/2 + (1/2)^n \leq x \leq 1. \end{cases}$$

18. Let X be a metric space and A a subset of X. A is said to be dense in X if $\bar{A} = X$. Prove that A is dense in $X \Leftrightarrow$ the only closed superset of A is $X \Leftrightarrow$ the only open set disjoint from A is $\emptyset \Leftrightarrow A$ intersects every non-empty open subset of X.

19. Prove that the set $\{m/2^n : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

20. Prove that $\frac{1}{4}$ is an element of the Cantor set.

21. Describe the boundary of each of the following subsets of the real line: the integers, the rationals; $[0,1]; (0,1)$. Do the same for each of the following subsets of the complex plane: $\{z : |z| < 1\}; \{z : |z| \leq 1\}; \{z : Im(z) > 0\}$.

22. Let (X, d) be a metric space. Then f as a function from X to $\mathbb{R}$ is continuous at $c \in X$ iff for every sequence $\{x_n\}$ converging to c, $f(x_n)$ converges.

23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is continuous at one point and f satisfies

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, then $f(x) = ax$ for some a.

24. Prove that $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

25. Let x_0 be a fixed point in a metric space (X, d) and to each point x in X, define a real valued function f_x on X by $f_x(y) = d(y, x) - d(y, x_0)$.
(a) Show that f_x is bounded.
(b) Show that f_x is continuous.
By (a) and (b), the mapping F defined by $F(x) = f_x$ is a mapping of X into $C(X, \mathbb{R})$.
(c) Show that F is an isometry.
F is thus an isometry of X into the complete metric space $C(X, \mathbb{R})$. We define the completion X^* of X to be the closure of $F(X)$ in $C(X, \mathbb{R})$.
(d) Show that X^* is a complete metric space that contains an isometric image of X.