Assignment sheet

The Fourier transform

---

1. Let $f \in \mathfrak{F}_a$ with $a > 0$, then show that $f^{(n)} \in \mathfrak{F}_b,\ \forall n \in \mathbb{N},\ 0 \le b < a$.

2. Compute the Fourier transform of (i) $e^{-\pi x^2}$, (ii) $\dfrac{1}{\pi}\dfrac{y}{y^2+x^2}$, (iii) $\operatorname{sech}(\pi x)$.

3. Suppose $f \in \mathfrak{F}_a,\ 0 < b < a, x, R \in \mathbb{R}$ and V is the line joining $R - ib,\ R + ib$, then find $\lim\limits_{R\to\infty} \displaystyle\int_V \frac{f(z)}{z-x}dz,\ \lim\limits_{R\to-\infty} \int_V \frac{f(z)}{z-x}dz$.

4. Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0,\ \forall \xi \in \mathbb{R}$. Then show that $f = 0$.

   Hint: Fix $t \in \mathbb{R}$. Set $A(z) = \displaystyle\int_{-\infty}^{t} f(x)e^{-2\pi iz(x-t)}dx,\ B(z) = -\int_{t}^{\infty} f(x)e^{-2\pi iz(x-t)}dx$, $F(z) = \begin{cases} A(z), & \operatorname{Im}(z) \ge 0, \\ B(z), & \operatorname{Im}(z) < 0. \end{cases}$ Show that F is entire, bounded function. Then show that $F \equiv 0$. From this deduce that $\displaystyle\int_{-\infty}^{t} f(x)dx = 0,\ \forall t \in \mathbb{R}$, and conclude $f \equiv 0$.

5. Verify the Fourier inversion formula for the functions given in Exercise 2.

6. If $A > 0,\ B \in \mathbb{R}$, then find $\displaystyle\int_0^\infty e^{-(A+iB)x}dx$.

7. If $g : \mathbb{R} \to \mathbb{C}$ be continuous, then show that $f(z) = \displaystyle\int_{-n}^{n} g(t)e^{2\pi itz}dt$ is entire.

8. Let f has moderate decrease, $a \in \mathbb{R}$, $t > 0$. Then prove that following:

   (i) If $g_1(x) = f(x)e^{2\pi iax}$, then $\hat{g}_1(\xi) = \hat{f}(\xi - a)$,

   (ii) If $g_2(x) = f(x - a)$, then $\hat{g}_2(\xi) = \hat{f}(\xi)e^{-2\pi ia\xi}$,

   (iii) If $g_3(x) = f(\frac{x}{t})$, then $\hat{g}_3(\xi) = t\hat{f}(t\xi)$.

9. Prove the following identities for $a \in \mathbb{R}$, $t > 0$.

   (i) $\displaystyle\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} \frac{e^{2\pi ian}}{\sqrt{t}}$,

   (ii) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi ian}}{\cosh(\frac{n\pi}{t})} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi t(n+a))}$,

   (iii) $\displaystyle\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = \coth(\pi a)$.

10. Prove or disprove: The function $f(z) = e^{z^2}$ satisfies the maximum modulus priciple in any unbounded domain.

11. Fix $\theta_1, \theta_2 \in (0, 2\pi)$ such that $\theta_2 - \theta_1 = \frac{\pi}{\beta}$, and $S = \{re^{i\theta} : r > 0, \theta_1 < \theta < \theta_2\}$. Suppose $F : \bar{S} \to \mathbb{C}$ is continuous, holomorphic on S, and $|F(z)| \leq 1$, $z \in \partial S$. If there exist $C, a > 0$, $\alpha \in (0, \beta)$ such that $|F(z)| \leq Ce^{a|z|^\alpha}$, $\forall z \in S$, then prove that $|F(z)| \leq 1$, $\forall z \in S$.
Hint: Use the technique which is employed the proof of the Phragmén-Lindelöf theorem.