PROBLEMS FOR WEEK-3

PROBLEM 1. Discuss the construction of Riemann sphere, in $\mathbb{R}^3$.

PROBLEM 2. Discuss real indefinite integration using using Cauchy residue formula.

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \pi,$

(2) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{ax}} = \frac{\pi}{sin(\pi a)}.$

PROBLEM 3. Let f be a function (need not be holomorphic) from $\mathbb{C}$ to the $\mathbb{C}\cup\{\infty\}$. Compute the Laurent series (if exists) representation of f in the following cases at center z_0 and find the annulus of convergence in each case:

(i) $f(z) = \frac{1}{1-z}$ at $z_0 = 0$.

(ii) $f = \dfrac{1}{1-z}$ at $z_0 = 1, 2$

(iii) $f(z) = \sqrt{z}$ at $z_0 = 0, 1$.

(iv) $f(z) = \sqrt{z(z-1)}$ at $z_0 = 0, 1$.

(v) $f = \sin(1/z)$ on $\Omega =$ at $z_0 = 0, 1/2\pi$.

Analyze the boundary behavior of each of the above function.

PROBLEM 4. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series centered at z_0 with radius δ. Then show that $f \equiv 0$ on $\mathbb{D}_\delta(z_0)$ if and only if $a_n = 0$ for each $n \in \mathbb{N}$.

PROBLEM 5. Let f be holomorphic function on any open set not containing zero then show that:

$$\frac{\partial f}{\partial r} + \frac{i}{r}\frac{\partial f}{\partial \theta} = 0.$$

PROBLEM 6. Every very domain (open, connected set) in $\mathbb{C}$ is path connected via piecewise C^1 curve.

PROBLEM 7. Let Ω be a domain in $\mathbb{C}$ and let Z be a discrete subset of Ω without any limit point contained in Ω then show that $\Omega \setminus Z$ is open-connected. Note the similar conclusion does not hold for open connected subsets of $\mathbb{R}$.

PROBLEM 8. Every non-trivial open subset of $\mathbb{C}$ has a non-empty boundary. Which subsets of a arbitrary topological space has empty boundary?

PROBLEM 9 (Cauchy Kernel). Let γ be a curve is in $\mathbb{C}$. Let ϕ be a continuous function on γ (i.e. image of γ) then

$$f(z) := \int_\gamma \frac{\phi(\zeta)}{\zeta - w} dz, \quad w \in \mathbb{C} \setminus \gamma$$

is holomorphic at w.

PROBLEM 10. Show that the unit circle $\mathbb{T}$ is the natural boundary for $\sum_{n=0}^{\infty} z^{n!}$, that is the above analytic function can not be extended in near any point on $\mathbb{T}$.

PROBLEM 11. Find examples of an essential singularity one isolated. Discuss singularity as branching point.

PROBLEM 12. Discuss contour integral around a isolated essential singularity of a meromorphic fucntion?

PROBLEM 13. Find singularities of the functions below. And identify the singularities (first tell if isolated or not, and then type):

(1) $\frac{a+bz}{c+dz}$ where $a, b, c, d \in \mathbb{R}$,
(2) $\sum_{n=0}^{\infty} (\frac{z}{1+z})^n$, also try to draw the area of convergence of this map,
(3) $\frac{z}{\sin(z)}$,
(4) $\sin(\frac{1}{1-z})$.

PROBLEM 14. How do we visualize complex (meromorphic functions)? We will discuss this in the last tutorial session.