Annual Foundation School (AFS)

Indian Institute of Technology Hyderabad
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Week 2-Assignment   Instructor: P. Muthukumar

Ref. Book: Complex Analysis by Elias M. Stein, Rami Shakarchi.

Notation: $\mathbb{R}, \mathbb{C}$ and $\mathbb{D}$ denote the set of all real numbers, the set of complex numbers and the open unit disk in the complex plane, respectively.

Problems:

1. Compute the integrals:
   (a) $\displaystyle\int\limits_{|z|=1} \frac{\text{Im}(z)e^z \cos z}{z}\, dz$ and $\int_C \frac{dz}{z^2(z^2+4)}$, where C consists of $|z| = 3$ counter clockwise and $|z| = 1$ clockwise.
   (b) $\displaystyle\int_{|z|=1} \frac{p(z)}{z^n}\, dz$, where $p(z)$ is a polynomial of degree n.
   (c) $\displaystyle\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta + \theta)\, d\theta$ Hint: consider e^z.
   (d) $\displaystyle\int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta$ Hint: $2\cos\theta = e^{i\theta} + e^{-i\theta}$.
   (e) $\displaystyle\int_0^{2\pi} e^{2\cos\theta} \cos(2\sin\theta - \theta)\, d\theta$ Hint: consider $\dfrac{e^{z^2}}{z^2}$.
   (f) $\displaystyle\int_0^{2\pi} e^{e^{i\theta} - in\theta}\, d\theta$, where n is an integer.
   (g) $\displaystyle\int_{|z|=1} |z - a|^{-4}\, |dz|$, where $a > 1$ is a constant.

2. Show that $\frac{1}{6\cdot 7\cdot 8} \int_{|z|=1} \frac{dz}{e^{-z}z^6} = \int_{|z|=1} \frac{dz}{e^{-z}z^9}$.

3. Questions $1 - 4$ in the Exercises of Chapter 2 in the reference book.

4. $\sum_{n=0}^{\infty} a_n(z - 1)^n$ be the power series expansion of the function $\frac{1}{z^2+1}$. Find the radius of convergence of the series.

5. What can you say about the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where (a_n) is the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$

6. Find all entire functions such that $|f(z)| \le 1 + 2\sqrt{|z|}$ for all $z \in \mathbb{C}$.
7. Let $f(z)$ be an entire function such that $|f(z)| \le K|z|$, for all $z \in \mathbb{C}$, for some $K > 0$. If $f(1) = i$, then find the value of $f(i)$.
8. Let f be an entire function which satisfies the following two equations

$$f(z+1) = f(z), \;\; f(z+2i) = f(z) \;\; \text{for every } z \text{ in } \mathbb{C}.$$

   Prove that f must be a constant function.
9. For any non-constant polynomial P, prove that $P(z) \to \infty$ as $z \to \infty$.
10. Find all entire functions f such that $f''(1/n) = e^{1/n}$ for all natural number $n \ge 1$.
11. If f is entire function such that $f(x) = f(ix)$ for all $x \in (0,1)$, should f be an even function? Hint: consider $f(z) - f(iz)$.
12. Let f be an entire function such that $f(x) \in \mathbb{R}$ and $f(ix) \in i\mathbb{R}$ for all $x \in \mathbb{R}$. Should f be an odd function? Hint: consider $-\overline{f(-\overline{z})}$.
13. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on $\mathbb{D}$. Show that $f(x) \in \mathbb{R}$ for all $x \in (-1,1)$ if and only if $a_n \in \mathbb{R}$ for all $n = 0, 1, 2, \ldots$.
14. Let D be a domain and let f, g be analytic functions on D such that $f(z)g(z) = 0$ for all $z \in D$. Prove that either $f \equiv 0$ or $g \equiv 0$ in D.
15. Questions $7, 9, 10$ in the Exercises of Chapter 2 in the reference book.
16. Prove that there is no sequence of polynomial converges to the function $1/z$ uniformly on $\partial\mathbb{D} = \{z : |z| = 1\}$. (Hint: Cauchy's Theorem)
17. Questions $13 - 15$ in the Exercises of Chapter 2 in the reference book.
18. Question 4 in the Problems of Chapter 2 in the reference book.

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