AFS-I, IIT Hyderabad
Problem Sheet (Complex Analysis)

Instructor : Prof. G. Ramesh (Week 1)

Problem 1. *Prove that a set* $\Omega \subset \mathbb{C}$ *is compact if and only if every sequence* $\{z_n\} \subset \Omega$ *has a subsequence that converges to a point in* Ω.

Problem 2. *An open covering of* Ω *is a family of open sets* U_α *(not necessarily countable) such that* $\Omega \subset \bigcup_\alpha U_\alpha$. *Prove that a set* Ω *is compact if and only if every open covering of* Ω *has a finite subcovering.*

Problem 3. *A set* Ω *is said to be pathwise connected if any two points in* Ω *can be joined by a (piecewise-smooth) curve entirely contained in* Ω.

(a) *Suppose first that* Ω *is open and pathwise connected, and that it can be written as* $\Omega = \Omega_1 \cup \Omega_2$ *where* Ω_1 *and* Ω_2 *are disjoint non-empty open sets. Choose two points* $w_1 \in \Omega_1$ *and* $w_2 \in \Omega_2$ *and let* γ *denote a curve in* Ω *joining* w_1 *to* w_2. *Consider a parametrization* $z : [0,1] \to \Omega$ *of this curve with* $z(0) = w_1$ *and* $z(1) = w_2$, *and let*

$$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s < t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$.

(b) *Conversely, suppose that* Ω *is open and connected. Fix a point* $w \in \Omega$ *and let* $\Omega_1 \subset \Omega$ *denote the set of all points that can be joined to* w *by a curve contained in* Ω. *Also, let* $\Omega_2 \subset \Omega$ *denote the set of all points that cannot be joined to* w *by a curve in* Ω. *Prove that both* Ω_1 *and* Ω_2 *are open, disjoint and their union is* Ω. *Finally, since* Ω_1 *is non-empty (why?) conclude that* $\Omega = \Omega_1$ *as desired.*

Problem 4. (a) *Let* z, w *be two complex numbers such that* $\overline{z}w \neq 1$. *Prove that:*

$$\left| \frac{w - z}{1 - \overline{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(b) *Prove that for a fixed* w *in the unit disc* $\mathbb{D}$, *the mapping*

$$F : z \to \frac{w - z}{1 - \overline{w}z}$$

satisfies the following conditions:

(a) F *maps the unit disc to itself (i.e.* $F : \mathbb{D} \to \mathbb{D}$*), and is holomorphic.*

(b) F *interchanges* 0 *and* ω, *namely* $F(0) = \omega$ *and* $F(\omega) = 0$.

(c) $|F(z)| = 1$ *if* $|z| = 1$.

(d) $F : \mathbb{D} \to \mathbb{D}$ *is bijective.*

Problem 5. *Suppose* U *and* V *are open sets in the complex plane. Prove that if* $f : U \to V$ *and* $g : V \to \mathbb{C}$ *are two functions that are differentiable (in the real sense, that is, as functions of the two real variables* x *and* y*), and* $h = g \circ f$, *then*

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

This is the complex version of the chain rule.

Problem 6. *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ *is the* ***Laplacian***

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Problem 7. *Consider the function defined by* $f(x+iy) = \sqrt{|x||y|}$, *whenever* $x, y \in \mathbb{R}$. *Show that* f *satisfies the Cauchy-Riemann equations at the origin, yet* f *is not holomorphic at 0.*

Problem 8. *Suppose that* f *is holomorphic in an open set* Ω. *Prove that in any one of the following cases:*

(a) Re(f) is constant;

(b) Im(f) is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

Problem 9. *Consider the function f defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is infinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0) = 0$ for all $n \geq 1$. Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for x near the origin.

Problem 10. *Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.*

Problem 11. *Let $f : G \to \mathbb{C}$ be a holomorphic function on an open set G. Define*

$$f^*(z) = \overline{f(\overline{z})}, \quad z \in G^* = \{\overline{z} : z \in G\}.$$

Show that f^* is also holomorphic on G^*.

Problem 12. *Let G be a region and suppose $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is a constant function.*

Problem 13. *Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = \exp(int)$, $\quad t \in [0, 2\pi]$, $n \in \mathbb{Z}$. Show that*

$$\int_\gamma \frac{1}{z}\, dz = 2\pi i n.$$

Problem 14. *Let γ be a closed polygon $[1-i, 1+i, -1+i, -1-i, 1-i]$. Find $\int_\gamma \frac{1}{z}\, dz$.*

Problem 15. *Show that if $|a| < r < |b|$, then*

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle at the origin of radius r with positive orientation.