Representation theory of finite groups

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Throughout, fix a $\mathbb{C}$-vector space V, and a finite group G.

1. Let $T, S \in \mathrm{End}(V)$ be two commuting operators. Prove that $\mathrm{Ker}(S)$ is T-invariant.

2. Fix a $\mathbb{C}$-vector space V and $T \in \mathrm{End}(V)$. Suppose $V = V_1 \oplus V_2$, and moreover both V_1, V_2 are T-invariant subspaces. Then find a basis of V such that matrix of T is block diagonal.

3. Fix $T \in \mathrm{End}(V)$. Prove for any fixed vector $v \in V$, that $\mathbb{C}[T] \cdot v := \operatorname{span}_{\mathbb{C}}\{v,\ Tv,\ T^{\circ 2}v,\ \ldots, T^{\circ n}v, \ldots\}$ is a T-invariant subspace; recall $\mathbb{C}[T]$ is the polynomial algebra on (symbol/operator) T.

4. Fix $T \in \mathrm{End}(V)$. For any subspace $W \leq V$, prove that:

   i) W is T-invariant iff W is $\mathbb{C}[T]$-invariant.

   ii) W is T-irreducible iff W is $\mathbb{C}[T]$-irreducible.

   iii) W is T-indecomposable iff W is $\mathbb{C}[T]$-indecomposable.

5. Recall, $T \in \mathrm{End}(V)$ is diagonalisable, iff the minimal polynomial of T has distinct roots.

6. Prove that every irreducible G-representation is also indecomposable.

7. Given a G-representation (ρ, V), verify that $\rho_{g^{-1}}$ equals $\left(\rho_g\right)^{-1}$.

8. Given two G-representations (ρ^V, V) and (ρ^W, W), check that $g \cdot (v, w) := \left(\rho^V_g(v),\ \rho^W_g(w)\right)$ defines a G-representation on $V \oplus W$.

9. Consider the set

$$\left\{T_i \;\middle|\; i \text{ is an element of the indexing set } I\right\},$$

where each T_i is an endomorphism in a finite-dimensional vector space V over the complex numbers $\mathbb{C}$. Prove that if the above family is commutative, then there exists a common eigenvector for each endomorphism T_i.

10. Find all the irreducible representations for $\mathbb{Z}_3$ up to isomorphism.

11. Prove that all the irreducible representations of an abelian group are one-dimensional.

1. Given two G-representations V, W, prove that $T : V \oplus W \longrightarrow W \oplus V$ defined by $(v, w) \xrightarrow{T} (w, v)$ is an isomorphism of G-representations.

2. For any fixed basis $\{v_1, \ldots, v_n\}$ of V, check that $v_i \mapsto e_i$ defines an isomorphism form V to $\mathbb{C}^n$.

3. Given a representation (V, ρ), and the transformation $T : V \longrightarrow \mathbb{C}^n$ as above. Prove that
$$\widetilde{\rho}_T : G \longrightarrow GL_n(\mathbb{C})$$
$$g \longrightarrow T\rho_g T^{-1}$$
defines a G-representation of $\mathbb{C}^n$.

4. Prove the following are equivalent for a G-representation V:
   a) V is completely reducible.
   b) Every proper sub-representation W in V has a compliment; i.e., there exists a sub-representation W' in V such that $V = W \oplus W'$.
   c) The short exact sequence $0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$ splits.

5. Let S_3 denote the permutation group on letters $\{1, 2, 3\}$. Consider the permutation representation $\rho^{\text{Perm}} : S_3 \longrightarrow GL_3(\mathbb{C})$, on $V = \mathbb{C}^3 = \mathbb{C}\{e_1,\ e_2,\ e_3\}$ (standard basis vectors), via:
$$\rho^{\text{Perm}}_{(123)} \longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \rho^{\text{Perm}}_{(12)} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Prove that $W = \mathbb{C}\{e_1 + e_2 + e_3\}$ is a sub-representation, and moreover V has a 2-dimensional irreducible sub-representation W' such that $V = W \oplus W'$.

6. Find the number of conjugacy classes in S_n.

7. Consider the regular representation of S_2. Decompose $\mathbb{C}[S_2]$ into direct sum of irreducible sub-representations.

8. Consider the regular representation of S_3. Find all the 1-dim. sub-representations of $\mathbb{C}[S_3]$. Next, if $\{e_\sigma \mid \sigma \in S_3\}$ is a basis for $\mathbb{C}[S_3]$, check that $v = \left[e_1 - e_{(12)}\right] \times \left[e_1 + e_{(13)}\right]$ generates an irreducible 2-dim. sub-representation. Thereby, decompose $\mathbb{C}[S_3]$ into direct sum of irreducible sub-representations.

9. Let $T : V \to V'$ be a G-invariant transformation (G-intertwiner) between two G-representations V and V'. Then prove that $\text{Ker}(T)$ and $\text{Im}(T)$ are also G- invariant Subspaces . Thereby, show that if V and V' are non-isomorphic irreducible representations, then $T \equiv 0$.
   In particular if $V = V'$ then show that $T = \lambda I$ for some $\lambda \in \mathbb{C}$.

10. In the above problem, show that the dimensions of the space of G-intertwiners $\text{Hom}_G(V, V')$, is 1 if $V \simeq V'$ and is 0 otherwise.

11. Let (ρ, V) be an irreducible representation of G then prove that the centre of G act as a scalar.

12. Compute the character table for S_3 and verify row orthogonality relation .