Annual Foundation School - I (2024), IIT Hyderabad

Module 3 - Linear Groups - Tutorial Sheet 1

Date and Time: December 16, 2024, 2:30 PM - 3:30 PM

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Q.1. Let $G = \mathrm{GL}_n$ and $X = \mathrm{M}_n$ over $\mathbb{R}$ or $\mathbb{C}$. For $P \in G$ and $A \in X$, prove that the operation

$$P \cdot A := (P^t)^{-1} A P^{-1}$$

defines a group action of G on X. The orthogonal group is defined as $\mathrm{O}_n = \{P \in \mathrm{GL}_n \mid P^t P = I\}$. Show that $\mathrm{O}_n = \mathrm{Stab}(I)$.

Q.2. Let $G = \mathrm{GL}_n(\mathbb{C})$ and $X = M_n(\mathbb{C})$. For $P \in G$ and $A \in X$, prove that the operation

$$P \cdot A := (P^*)^{-1} A P^{-1}$$

defines a group action of G on X.

(i) The unitary group is defined as

$$\mathrm{U}_n = \{P \in \mathrm{GL}_n(\mathbb{C}) \mid P^* P = I\}.$$

Show that $\mathrm{U}_n = \mathrm{Stab}(I_n)$.

(ii) Show that $\mathrm{U}_1 = S^1 = \mathrm{SO}_2(\mathbb{R})$.

Q.3. Find a subgroup of $\mathrm{GL}_2(\mathbb{R})$ that is isomorphic to $\mathbb{C}^\times$.

Q.4. Prove that for every $n \in \mathbb{N}$, $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to a subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$.

Q.5. Identify a group G that is isomorphic to $\mathrm{GL}_n^+(\mathbb{R})$.

Q.6. Show that $\mathrm{SO}_2(\mathbb{C})$ is not a bounded set in $\mathbb{C}^4$.

Revision from the Lecture Notes.

Q.7. Using the standard real inner product on $\mathbb{R}^n$, deduce that for any $n \times n$ matrix P,

$$\langle v, w \rangle = \langle Pv, Pw \rangle \implies P^t P = I_n.$$

Prove that $\mathrm{O}_n(\mathbb{R})$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Q.8. Using the standard Hermitian inner product on $\mathbb{C}^n$, deduce that for any $n \times n$ matrix P,

$$\langle v, w\rangle = \langle Pv, Pw\rangle \implies \overline{P^t}P = I_n.$$

Prove that $\mathrm{U}_n(\mathbb{C})$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

2. Additional Problems

Q.9. Show that for any $n \times n$ matrices A and B over $\mathbb{R}$, one has $\det(AB) = \det(A) \cdot \det(B)$.

Q.10. Let H be the linear groups,

$$SL_n(\mathbb{R}),\ SL_n(\mathbb{C}),\ \mathrm{SO}_2(\mathbb{R}),\ \mathrm{SO}_2(\mathbb{C}),\ \mathrm{SU}_2(\mathbb{C}),\ \mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{C}).$$

Answer the following for each linear group H and make a table with Yes/No.

(i) Is H connected?

(ii) Is H path-connected?

(iii) Is H simply connected?

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Module 3 - Linear Groups - Tutorial Sheet 2

Date and Time: December 18, 2024, 2:30 PM - 3:30 PM

Dr. Neeraj Kumar, Department of Mathematics, IIT Hyderabad

Q.1. Let A be an $n \times n$ skew-symmetric matrix with variable entries x_{ij}. Prove that:

(i) If n is odd, then $\det(A) = 0$.

(ii) If $n = 2, 4$, then $\det(A)$ is a square of some polynomial expression.

Q.2. Let $G = \mathrm{GL}_n$ and $X = \mathrm{M}_n$ over $\mathbb{R}$ or $\mathbb{C}$. For $P \in G$ and $A \in X$, prove that the operation

$$P \cdot A := (P^t)^{-1} A P^{-1}$$

defines a group action of G on X.

(i) Show that $\mathrm{Sp}_{2n} = \mathrm{Stab}(J)$, where J is the $2n \times 2n$ skew-symmetric block matrix:

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

(ii) Show that $\mathrm{O}_{3,1} = \mathrm{Stab}(I_{3,1})$, where $I_{3,1}$ is the 4×4 diagonal matrix:

$$I_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Q.3. Prove that the following matrices are symplectic if the blocks are $n \times n$:

$$\begin{pmatrix} & -I \\ I & \end{pmatrix}, \quad \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad \begin{pmatrix} I & B \\ 0 & I \end{pmatrix},$$

where $B = B^t$ and A is invertible.

Q.4. Prove that $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, but $\mathrm{Sp}_4(\mathbb{R}) \neq \mathrm{SL}_4(\mathbb{R})$.

Q.5. Let G be a group, and $C(a)$ denote the conjugacy class of an element $a \in G$.

(i) Compute $C(a)$ for $G = S_3$ and $a = e, (1,2), (1,2,3)$.

(ii) Show that if $g \in Z(G)$, then $C(g) = \{g\}$.

(iii) Compute $Z(\mathrm{SU}_2)$, the center of SU_2.

Q.6. Answer the following:

(i) Is $O_n(\mathbb{R})$ a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$?

(ii) Is $\mathrm{SL}_n(\mathbb{R})$ a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$?

(iii) Is $\mathrm{Sp}_{2n}(\mathbb{R})$ a normal subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$?

(iv) Is U_n a normal subgroup of $\mathrm{GL}_n(\mathbb{C})$?

Q.7. Show that the eigenvalues of a unitary matrix lie on the unit circle in the complex plane.

2. Additional Problems

Q.8. Let $H \in \{\mathrm{O}_n, \mathrm{U}_n, \mathrm{Sp}_{2n}\}$ be a subgroup of the appropriate general linear group. Answer the following:

(i) Is H an open subset of the general linear group?

(ii) Is H a closed subset of the general linear group?

(iii) Is H compact?

(iv) Is H connected?

Q.9. Try rewriting the proof of the theorem discussed in the lecture.

Theorem. *The conjugacy classes of* SU_2 *are precisely the latitudes* $\boldsymbol{Lat}_c$, *where* $-1 \leq c \leq 1$, *defined as:*

$$\mathbf{Lat}_c = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; x_1 = c \text{ and } x_2^2 + x_3^2 + x_4^2 = 1 - c^2\right\}.$$

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Module 3 - Linear Groups - Tutorial Sheet 3

Date and Time: December 19, 2024, 4:00 PM - 5:00 PM

Dr. Neeraj Kumar, Department of Mathematics, IIT Hyderabad

Q.1. Prove the following basic properties:

(i) If v is an eigenvector of A with eigenvalue λ, then v is also an eigenvector of e^A with eigenvalue e^λ.

(ii) Given the exponential matrix e^A, prove that conjugation by an invertible matrix P satisfies $P^{-1}e^AP = e^{P^{-1}AP}$.

(iii) Let A and B be $n \times n$ matrices such that $AB = BA$, then $e^{A+B} = e^Ae^B$.

(iv) For any $A \in \mathrm{M}_n$, prove that e^A is an invertible matrix.

(v) For any $A \in \mathrm{M}_n(\mathbb{C})$, prove that $\det(e^A) = e^{\mathrm{trace}(A)}$.

Q.2. Determine the differentiable homomorphisms from $(\mathbb{C}, +)$ to $\mathrm{SL}_n(\mathbb{C})$.

Q.3. Describe all one-parameter subgroups of $(\mathbb{C} - \{0\}, \cdot)$.

Q.4. Describe by equations the image of all one-parameter subgroups of the group of real 2×2 invertible diagonal matrices.
(Remark: Further, extend this to $n \times n$ invertible diagonal matrices (real or complex)).

Q.5. Let $\varphi : (\mathbb{R}, +) \to \mathrm{GL}_n(\mathbb{R})$ be a one-parameter subgroup. Prove that $\ker(\varphi)$ is either trivial, or the whole group, or else it is infinite cyclic.

Q.6. Find the conditions on a matrix A such that e^{tA} is a one-parameter subgroup of the special unitary group SU_n.

Q.7. Determine the one-parameter subgroups of U_2. More generally, extend this to U_n for any n.

Q.8. If $A = \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}$, then prove that the matrix exponential e^{tA} is given by:

$$\begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}.$$

Using this, show that the map $\varphi : (\mathbb{R}, +) \longrightarrow GL_2(\mathbb{R})$ defined by mapping t to e^{tA} is not an injective group homomorphism.

Q.9. Let $\varphi(t) = e^{tA}$ be a one-parameter subgroup of a group G. Prove that the cosets of $\operatorname{im}(\varphi)$ are matrix solutions of the differential equation $\frac{dX}{dt} = AX$.

2. Revision

Definition. Let G be a subgroup of GL_n. A one-parameter subgroup in G is a one-parameter subgroup $\varphi(t) = e^{tA}$ in GL_n such that $\varphi(t) \in G$ for all $t \in \mathbb{R}$.

Revision. Let $G \in \{\mathrm{O}_n(\mathbb{R}), \mathrm{U}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{R}), \mathrm{SU}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{R}), \mathrm{SL}_n(\mathbb{C})\}$ be a subgroup of the appropriate general linear group. Then, in each case, find conditions on the matrix A such that $e^{tA} \in G$ for all $t \in \mathbb{R}$ if and only if A satisfies the required conditions. Make a table for the above.

Table: Conditions on A

Group G	**Condition on** A
$\mathrm{O}_n(\mathbb{R})$	$A^T + A = 0$
$\mathrm{U}_n(\mathbb{C})$	$A^* + A = 0$
$\mathrm{SO}_n(\mathbb{R})$	$A^T + A = 0,\ \operatorname{trace}(A) = 0$
$\mathrm{SU}_n(\mathbb{C})$	$A^* + A = 0,\ \operatorname{trace}(A) = 0$
$\mathrm{SL}_n(\mathbb{R})$	$\operatorname{trace}(A) = 0$
$\mathrm{SL}_n(\mathbb{C})$	$\operatorname{trace}(A) = 0$

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Module 3 - Linear Groups - Tutorial Sheet 4

Date and Time: December 20, 2024, 4:00 PM - 5:00 PM

Dr. Neeraj Kumar, Department of Mathematics, IIT Hyderabad

Q.1. Answer the following:

(i) Find $(A + B\epsilon)^{-1}$, where A is an invertible matrix. (Refer to textbook or lecture notes for the introduction of an infinitesimal element ϵ satisfying $\epsilon^2 = 0$)

(ii) Prove that $\det(I + A\epsilon) = 1 + (\text{trace}(A))\epsilon$.

(iii) Compute $\det(A + B\epsilon)$, where A is an invertible matrix.

(iv) Compute the infinitesimal tangent vectors to the plane curve $y^2 = x^3$ at the points $(1, 1)$.

(v) Consider the group

$$G = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R}^\times \right\},$$

where $\mathbb{R}^\times$ denotes the multiplicative group of nonzero real numbers. Show that the subset

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in G : xy = 1 \right\}$$

is a subgroup of G, and compute its Lie algebra.

Summary of Lie Algebras of Some Classical Linear Groups

Linear Group G	**Lie Algebra,** Lie(**G**)	dim Lie(G)	**Defining Condition for** A
$\mathrm{GL}_n(\mathbb{R})$	$\mathfrak{gl}_n$	n^2	None (all $n \times n$ matrices)
$\mathrm{GL}_n(\mathbb{C})$	$\mathfrak{gl}_n(\mathbb{C})$	$2n^2$	None (all $n \times n$ complex matrices)
$\mathrm{SL}_n(\mathbb{R})$	$\mathfrak{sl}_n$	$n^2 - 1$	$\text{trace}(A) = 0$
$\mathrm{SL}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$2(n^2 - 1)$	$\text{trace}(A) = 0$
$O_n(\mathbb{R})$	$\mathfrak{o}_n$	$\frac{n(n-1)}{2}$	$A^T = -A$ (skew-symmetric matrices)
$\mathrm{SO}_n(\mathbb{R})$	$\mathfrak{so}_n$	$\frac{n(n-1)}{2}$	$A^T = -A$ (skew-symmetric matrices)
U_n	$\mathfrak{u}_n$	n^2	$A^* = -A$ (skew-Hermitian matrices)
SU_n	$\mathfrak{su}_n$	$n^2 - 1$	$A^* = -A$, $\text{trace}(A) = 0$
$\mathrm{Sp}_{2n}(\mathbb{R})$	$\mathfrak{sp}_{2n}$	$n(2n + 1)$	$A^T J + JA = 0$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

Q.2. Use three different definitions (approaches) discussed in the lecture to find Lie(G).