NCM Annual Foundational School-I

IIT Hyderabad, December 2024

Exercise Set 1 : Review of Linear Algebra

[Note: In the following, m, n denote positive integers. For a finite set S, by $|S|$ we denote the cardinality of S. As usual $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively.]

1. Use Gaussian elimination to prove that if $AX = 0$ is a homogeneous system of linear equations, where A is an $m \times n$ matrix (with entries in a field K) and X is an $n \times 1$ column vector of variables, and if $m < n$, then the system has a nontrivial soluition.

2. Let V be a vector space over a field K, and suppose S and L are finite subsets of V such that S spans V and L is linearly independent. Prove that $|S| \geq |L|$.

3. Let V be a vector space over a field K such that V is finite-dimensional (which means that V contains a finite subset which spans V). Define what is meant by a *basis* of V. Show that V has a basis and any two bases of V have the same cardinality (called the *dimension* of V, and denoted by $\dim_K V$, or simply, $\dim V$). Further show that any linearly independent subset of V can be extended to a basis of V and also that any spanning subset of V can be contracted to a basis of V, i.e., it contains a subset which is a basis of V.

4. Let V be a finite-dimensional vector space over a field K, and let B be a subset of V. Show that the following conditions are equivalent.

   (a) B is a basis of V
   (b) B is a maximal linearly independent subset of V
   (c) B is a minimal spanning subset of V.

   Deduce that if $\dim V = n$ and if B contains n elements and B is linearly independent or B spans V, then B is a basis of V.

5. Let A be an $n \times n$ matrix with entries in a field K. Write doen as many definitions as you know of $\det A$, the determinant of A.

6. Let A be an $m \times n$ matrix with entries in a field K. Write doen as many definitions as you know of rank of A.

7. Define what is meant by an *elementary matrix*. How are elementary matrices related to (i) elementary row operations?, (ii) elementary column operations, and (iii) invertible matrices?

8. Let K be a field and let $X = M_n(K)$ be the set of all $n \times n$ matrices with entries in K. and $G = \mathrm{GL}_n(K)$ the group of all $n \times n$ nonsingular matrices with entries in K. Consider the action of G on X defined by left multiplication: $(g, A) \mapsto gA$. Describe the orbits of this action. In case $n = 3$ and $K = \mathbb{F}_2$ is the field with two elements, then write down all the orbits of the action and verify the class equation.

9. Consider the action of $G = \mathrm{GL}_n(\mathbb{C})$ on itself by conjugation: $(g, A) \mapsto gAg^{-1}$. What can you say about the orbits of this action?

10. Consider the action of $G = \mathrm{GL}_m(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ on the set X of all $m \times n$ matrices with entries in $\mathbb{R}$, defined by $(P, Q)A = Q^{-1}AP$ for $P \in \mathrm{GL}_m(\mathbb{R})$, $Q \in \mathrm{GL}_n(\mathbb{R})$ and $A \in X$. Verify that this is a group action. What can you say about the orbits of this action? Is it true that there are only finitely many orbits? If so, find the number of distinct orbits.

Bonus Problems

1. Show the the different definitions of determinant in your answer of Q. 5 are equivalent.

2. Show the the different definitions of rank in your answer of Q. 6 are equivalent.

3. Show that the reduced row echelon form of a matrix is unique.

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Exercise Set 2 : Bilinear Forms over Real Vector Spaces

[Note: In the following, n denotes a positive integer and $\mathbb{R}$ denotes the fields of real numbers.]

1. Let V be a real vector space of dimension n and let $\langle\ ,\ \rangle$ be a bilinear form on V. If $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_n)$ are two (ordered) bases of V, and if A and A' are matrices of $\langle\ ,\ \rangle$ with respect to B and B' respectively, then how are A and A' related with each other?

2. If A and A' are as in Q. 1 above, then is it true that the $\text{rank}(A) = \text{rank}(A')$? And is it true that A and A' have the same eigenvalues? Justify your answer.

3. Let V be a real vector space of dimension n and let $\langle\ ,\ \rangle$ be a bilinear form on V. We say that $\langle\ ,\ \rangle$ is *skew-symmetric* if $\langle v, w\rangle = -\langle w, v\rangle$ for all $v, w \in V$. Show that $\langle\ ,\ \rangle$ is skew-symmetric if and only if $\langle v, v\rangle = 0$ for all $v \in V$.

4. Show that every bilinear form on a finite-dimensional real vector space is a sum of a symmetric bilinear form and a skew-symmetric bilinear form.

5. Let V be a real vector space of dimension n and let $\langle\ ,\ \rangle$ be a symmetric bilinear form on V. Then the map $q : V \to \mathbb{R}$ defined by $q(v) = \langle v, v\rangle$ for $v \in V$, is called a quadratic form on V corresponding to $\langle\ ,\ \rangle$. Show that the bilinear form $\langle\ ,\ \rangle$ is determined by the quadratic form q., i.e., for any $v, w \in V$, we can write $\langle v, w\rangle$ in terms of suitable values of q.

6. Suppose P is an $n \times n$ matrix with entries in $\mathbb{R}$ and $A = P^tP$, where P^t denotes the transpose of P. Is the blinear form $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ defined by $\langle X, Y\rangle = X^tAY$ symmetric? And is it positive definite?

7. Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, where $a, b, d \in \mathbb{R}$. Show that the blinear form $\langle\ ,\ \rangle$ on $\mathbb{R}^2$ defined by $\langle X, Y\rangle = X^tAY$ is positive definite if and only if $a > 0$ and $ad - b^2 > 0$.

8. Let $\mathcal{P}_n$ be the vector space of polynomials in one variable of degree $\leq n$ with coefficients in $\mathbb{R}$. Show that
$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \quad \text{for } f, g \in \mathcal{P}_n$$
defines a symmetric bilinear form on $\mathcal{P}_n$. Is it positive definite? Write down a matrix of this bilinear form with respect to a basis of $\mathcal{P}_n$ of your choice when $n = 2$.

9. Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with entries in $\mathbb{R}$. Write down a basis of V and determine the dimension of V. Further show that $\langle A, B\rangle = \text{trace}(A^tB)$ for $A, B \in V$, defines a bilinear form on V. Is it a symmetric bilinear form? And is it positive definite? When $n = 2$, determine the matrix of this bilinear form with respect to the basis of V that you have written down earlier.

10. Let V be a real vector space of dimension n. Show that the set $\text{Bil}(V)$ of all bilinear forms on V forms a vector space over $\mathbb{R}$ with respect to pointwise addition and scalar multiplication. Is $\text{Bil}(V)$ finite-dimensional? If yes, then what is its dimension?

Bonus Problems

B1. Let A be a real symmetric matrix of size $n \times n$. Show the A is positive definite (i.e., the corresponding bilinear form on $\mathbb{R}^n$ is positive definite) if and only if the leading principal minors of A are positive, i.e., $\Delta_k > 0$ for $k = 1, \ldots, n$, where Δ_k denotes the determinant of the $k \times k$ submatrix of A formed by the first k rows and the first k columns of A.

B2. Let A be a real symmetric matrix of size $n \times n$. Then A (or the corresponding bilinear form on $\mathbb{R}^n$) is said to be *negative definite*) if $X^t AX < 0$ for all $X \in \mathbb{R}^n$ with $X \neq 0$. Is there a characterization of negative definite matrices similar to that in Q. B1? If yes, state and prove it.

B3. Let A be a real symmetric matrix of size $n \times n$. Then A (or the corresponding bilinear form on $\mathbb{R}^n$) is said to be *positive semi-definite* (or *nonnegative definite*) if $X^t AX \geq 0$ for all $X \in \mathbb{R}^n$. Is there a characterization of positive semi-definite matrices similar to that in Q. B1? If yes, state and prove it.

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Exercise Set 3 : Hermitian Forms and Orthogonality

[Note: In the following, m, n denote positive integers and as usual, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. Also $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ denote the sets of all $n \times n$ matrices with entries in $\mathbb{R}$ and $\mathbb{C}$, respectively.]

1. If $A \in M_n(\mathbb{C})$ is Hermitian, i.e., if $A^* = A$, then show that the eigenvalues of A are real numbers. Deduce that the trace and determinant of A are real numbers. Also deduce that the eigenvalues of a real symmetric matrix are real.

2. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and for $1 \leq j \leq n$, let e_j denote the vector in $\mathbb{C}^n$ with 1 in j-th position and 0 elsewhere. Compute $X^t AX$ if $X = e_i - e_j$ where $1 \leq i \leq j \leq n$.

3. Show that if $A \in M_n(\mathbb{R})$ is symmetric and positive definite, then the maximal entries of A are on its diagonal, i.e., $\max\{a_{ij} : 1 \leq i, j \leq n\} = \max\{a_{ii} : i = 1, \ldots, n\}$.

4. If $A \in M_n(\mathbb{C})$ is unitary, i.e., if $A^* = A^{-1}$, then show that $\langle AX, AY \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathbb{C}^n$, where $\langle\ ,\ \rangle$ denotes the standard dot product on $\mathbb{C}^n$. Deduce that eigenvalues of A in $\mathbb{C}$ have absolute value 1. Show that similar conclusions hold if $A \in M_n(\mathbb{R})$ is an orthogonal matrix, i.e., if $A^t = A^{-1}$.

5. If $A \in M_n(\mathbb{C})$ is a Hermitian matrix and $X \in \mathbb{C}^n$ is such that $A^m X = 0$ for some positive integer m, then show that $AX = 0$.

6. Let V be a complex vector space of dimension n and let $\langle\ ,\ \rangle$ be a positive definite Hermitian form on V. A subset $\{v_1, \ldots, v_m\}$ of V is said to be *orthogonal* if $\langle v_i, v_j \rangle = 0$ for all $i, j = 1, \ldots, m$ with $i \neq j$. Show that if $\{v_1, \ldots, v_m\}$ is an orthogonal set of nonzero vectors in V, then $\{v_1, \ldots, v_m\}$ is linearly independent. Deduce that if $\{v_1, \ldots, v_m\}$ is an *orthonormal* set, i.e., $\{v_1, \ldots, v_m\}$ is orthogonal and moreover, $\langle c_i, v_i \rangle = 1$ for each $i = 1, \ldots, n$, then $\{v_1, \ldots, v_m\}$ is linearly independent.

7. Let V be a complex vector space of dimension n and let $\langle\ ,\ \rangle$ be a Hermitian form on V. Show that if $\langle\ ,\ \rangle$ is positive definite, then it is nondegenerate on every subspace W of V.

8. If $P \in M_n(\mathbb{C})$ is is Hermitian, then show that eigenvectors corresponding to distinct eigenvalues of A are orthogonal (with respect to the standard dot product on $\mathbb{C}^n$).

9. Let A be an $m \times n$ matrix with entries in $\mathbb{R}$ and let $B = A^t A$. Show that B is positive semi-definite, i.e., $X^T BX \geq 0$ for all $X \in \mathbb{R}^n$. Further show that $\text{rank}(A) = \text{rank}(B)$.

10. Suppose $A \in M_n(\mathbb{C})$ is *unitarily diagonalizable*, i.e., there is a unitary matrix $P \in M_n(\mathbb{C})$ such that $P^* AP$ is a diagonal matrix. Show that A must be *normal*, i.e., $AA^* = A^* A$.

Bonus Problems

B1. Show that every $A \in M_n(\mathbb{C})$ is *unitarily triangularizable*, i.e., there exists a unitary matrix $P \in M_n(\mathbb{C})$ such that $P^{-1} AP$ is an upper triangular matrix.

B2. Let $A \in M_n(\mathbb{C})$. Show that A is a diagonal matrix if and only if A is upper triangular and normal.

B3. Show that if $A \in M_n(\mathbb{C})$ is normal, then it is *unitarily diagonalizable*, i.e., there exists a unitary matrix $P \in M_n(\mathbb{C})$ such that $P^{-1} AP$ is an diagonal matrix.

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Exercise Set 4 : Orthogonal Bases, Spectral Theorem, Conics and Quadrics

[Note: In the following, n denote positive integers and $\mathbb{K}$ denotes the field $\mathbb{C}$ of complex numbers, or the field $\mathbb{R}$ of real numbers. Also V denotes a vector space of dimension n over $\mathbb{K}$ and $\langle\ ,\ \rangle$ be a bilinear form on V which is Hermitian if $\mathbb{K} = \mathbb{C}$ and symmetric if $\mathbb{K} = \mathbb{R}$.]

1. Assume that $\langle\ ,\ \rangle$ is positive definite. For $v \in V$, define $\|v\| = \sqrt{\langle v, v\rangle}$. Let W be a subspace of V and let $\pi : V \to W$ denote the orthogonal projection of V onto W. Show that for any $v \in V$, the best approximation of v in W is $\pi(w)$, in the sense that $\|v - \pi(v)\| \leq \|v - w\|$ for all $w \in W$.

2. (Riesz Representation Theorem) Assume that $\langle\ ,\ \rangle$ is positive definite. Suppose f is is any linear functional on V, i.e., f : is a linear map from V to $\mathbb{K}$. Show that there exists unique $y \in V$ such that $f(v) = \langle v, y\rangle$ for all $v \in V$.

3. A subspace W of V is said to be a *positive definite subspace* (w.r.t. $\langle\ ,\ \rangle$) if the restricted form $\langle\ ,\ \rangle\,|_{W\times W}$ is positive definite. Likewise, a sunspace W of V is said to be a *negative definite subspace* (w.r.t. $\langle\ ,\ \rangle$) if the restricted form $\langle\ ,\ \rangle\,|_{W\times W}$ is negative definite. Show that $V = V_+ \oplus V_- \oplus V_0$, where V_+ is a positive definite subspace of V, V_- is a negative definite subspace of V, and V_0 is a subspace of V on which the form $\langle\ ,\ \rangle$ is identically zero.

   (i) Show that V has a positive definite subspace V_+, a negative definite subspace V_+, and a subspace V_0 on which the form $\langle\ ,\ \rangle$ is identically zero., such that $V = V_+ \oplus V_- \oplus V_0$.

   (ii) Show that if V_+, V_- and V_0 are as in (i) above, then V_0 must be the null space $V^\perp$ of V.

   (iii) Show that if V_+, V_- and V_0 are as in (i) above, and if V_+', V_-' and V_0' also satisfy the properties in (i) above, then $\dim V_+ = \dim V_+'$, $\dim V_- = \dim V_-'$, and $\dim V_0 = \dim V_0'$.

   (iv) Show that the index (p, q, s) of the bilinear form $\langle\ ,\ \rangle$ is unique. In other words, if there are orthogonal bases with respect to which the matrices of $\langle\ ,\ \rangle$ are of the form

   $$\begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_{p'} & & \\ & -I_{q'} & \\ & & 0_{s'} \end{pmatrix}$$

   then $p = p'$, $q = q'$ and $s = s'$. [This is known as Sylvester's Law of Inertia.]

4. Find an orthogonal basis for the symmetric bilinear form X^tAY on $\mathbb{R}^n$ when

   $$\text{(i) } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{(ii) } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

5. (Gram-Schmidt Process) Assume that $\langle\ ,\ \rangle$ is positive definite. Given an arbitrary basis $(v_1, \ldots, v_n)$ of V, define$(w_1, \ldots, w_n)$ recursively as follows.

   $$w_1 = \frac{v_1}{\|v_1\|} \quad \text{and if } k > 1 \text{, then } w_k \text{ is obtained from } w_1, \ldots, w_{k-1} \text{ by} \quad w_k = \frac{v_k - \pi(v_k)}{\|v_k - \pi(v_k)\|},$$

   where $\pi(v_k) = \langle w_1, v_k\rangle w_1 + \cdots + \langle w_{k-1}, v_k\rangle w_{k-1}$ is the image of v_k under the projection of V onto the subspace spanned by $w_1, \ldots, w_{k-1}$. Show that $(w_1, \ldots, w_n)$ is an orthonormal basis of V.

6. Apply to the Gram-Schmidt Process to the basis (v_1, v_2, v_3) of $\mathbb{R}^3$, where

   $$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, .$$

7. Suppose $V = M_n(\mathbb{C})$ and $\langle A, B\rangle = \text{trace}(A^*B)$. Show that $\langle\ ,\ \rangle$ is Hermitian and find its signature.

8. Suppose $V = M_n(\mathbb{R})$ and $\langle A, B\rangle = \text{trace}(A^tB)$. Let W be the subspace of V consisting of skew-symmetric matrices, and let $\pi : V \to W$ be the orthogonal projection of V onto W, w.r.t. the form $\langle\ ,\ \rangle$. Compute $\pi(A)$, where $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}$.

9. Show that if $A, B \in M_n(\mathbb{C})$ are normal matrices such that $AB = BA$, then they can be simultaneously unitarily diagonalized, i.e., there exists a unitary matrix $P \in \text{GL}_n(\mathbb{C})$ such that P^*AP and P^*BP are diagonal matrices. Is the converse true? That is if $A, B \in M_n(\mathbb{C})$ are normal matrices that are simultaneously unitarily diagonalizable, then is it true that $AB = BA$?

10. Consider the conic in $\mathbb{R}^2$ given by $2x^2 + 4xy + 5y^2 + 4x + 13y - \frac{1}{4} = 0$. Diagonalize the symmetric matrix of the associated quadratic form, and use it to etermine if this conic is an ellpise, parabola, or hyperbola, or a degenerate conic. In a similar manner, determine the type of the quadric in $\mathbb{R}^3$ given by $x^2 + 4xy + 2xz + z^2 + 3x + z6 = 0$.

Bonus Problems

B1. Determine which of the results proved in the lectures for symmetric bilinear forms over $\mathbb{R}$ are valid for symmetric bilinear forms over an arbitray field K.

B2. Let A be a rectangular matrix of size $m \times n$ with entries in $\mathbb{R}$. Show that the eigenvalues of A^tA and AA^t are nonnegative real numbers and they have the same nonzero eigenvalues, i.e., if $\lambda \in \mathbb{R}$ with $\lambda > 0$, then

$$\lambda \text{ is an eigenvalue of } A^tA \iff \lambda \text{ is an eigenvalue of } AA^t.$$

B3. (Singular Value Decomposition) Let A be a rectangular matrix of size $m \times n$ with entries in $\mathbb{R}$. Show that there exist orthogonal matrices U and V of sizes $m \times m$ and $m \times n$, respectively, and an $m \times n$ diagonal matrix Σ with nonnegative entires on the diagonal such that $A = U\Sigma V^t$.