Group Theory Problem Set

AFS 2024 - IIT Hyderabad

Pradipto Banerjee

Note: Unless otherwise specified, all groups discussed here are finite, and all sets considered are nonempty and finite.

---

1. Group Actions and their Applications

1. Let P be a p-group for some prime p, acting on a set X, where $p \nmid |X|$. Prove that P has a *fixed point* $x \in X$; that is, there is an $x \in X$ such that $gx = x$ for all $g \in P$. Deduce that $Z(P) \neq \{1\}$ for any p-group.

2. Which of the following represent(s) the Class Equation of a group of order 10?

$$1+1+1+2+5, \quad 1+2+2+5, \quad 1+2+3+4, \quad 1+1+2+2+2+2.$$

3. Prove that if Q_8 acts *faithfully* on X, then $|X| \geq 8$.

4. Determine all groups which contain at most three conjugacy classes.

5. Determine the possible Class Equations for groups of order 8.

6. Classify all groups of order 8.

7. Let G be a group acting *transitively* on a set X. Prove that

$$\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)| = 1,$$

where $\mathrm{Fix}(g) = \{x \in X : gx = x\}$, the set of fixed points of g.

8. Let G be a nontrivial group acting *transitively* on a set X of cardinality at least 3. Show that $\mathrm{Fix}(g) = \emptyset$ for some $g \in G$. Deduce that if G is nontrivial and H is a proper subgroup of G, then $G \neq \bigcup_{g\in G} gHg^{-1}$.

9. Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian, and hence, $G = Z(G)$.

10. Let P be a p-group acting on X. Let $\mathrm{Fix}(G) = \{x \in X : gx = x \ \forall\ g \in G\}$. Prove that $|X| \equiv \mathrm{Fix}(G) \pmod{p}$.

---

2. Free Groups and presentatations

1. Let $a, b \in G$, where G is a group. Show that $\langle a, b\rangle = \langle bab^2, bab^3\rangle$

2. Prove that the group $G := \langle a, b, c \mid bab = c^2\rangle$ is *free*.

3. A subgroup H of G is referred to as *charcteristic* if for every automorphism σ of G, one has $\sigma(H) = H$ (e.g. $H = \{1\}$ and $H = G$).

   (a) Prove that every characteristic subgroup of G is normal.

   (b) Prove that $Z(G)$ is a characteristic subgroup of G.

4. Let G' denote the commutator of a group G.

   (a) Prove that G' is a characteristic subgroup of G.

   (b) Prove that G/G' is abelian.

   (c) Let N be a normal subgroup of G. Then G/N is abelian if and only if $G' \subset N$.

5. Show that the commutator subgroup of the free group $\mathscr{F}(a, b)$ is the *normal* subgroup generated by $aba^{-1}b^{-1}$. (The normal subgroup generated by $x \in G$ is the subgroup generated by the set $\{gxg^{-1} : g \in G\}$. It is the *smallest* normal subgroup of G containing x).

6. Consider the two groups

$$G_1 = \langle a, b \mid a^3 = 1 = b^7, aba^{-1} = b^2\rangle, \quad G_2 = \langle x, y \mid x^3 = 1 = y^7, xyx^{-1} = y^3\rangle.$$

Prove that $G_1 \cong G_2$.

7. Let $\phi : G \to G_1$ be a surjective group homomorphism. Let $S \subset G$ be such that $G_1 = \langle \phi(S)\rangle$. Further, suppose that $\ker \phi = \langle T\rangle$ where $T \subset G$. Show that $G = \langle S \cup T\rangle$.

8. Classify up to isomorphism all groups of order 55 by giving their presentations (Sylow theorems needed).

9. Determine the presentations for the Klein four-group and A_4.

10. Let A be a *finite* abelian group generated by two elements. Show that A is a direct sum of two cyclic groups.

---

3. Sylow's Theorems

1. Let G be a group of order 225 with a cyclic 5-Sylow subgroup. Show that G is abelian.

2. Let G be a group of order 385. Show that the 11-Sylow subgroup of G is normal and that the 7-Sylow subgroup is contained in $Z(G)$.

3. If $|G| = 108$, then show that G has normal subgroups of order 9 *or* 27.

4. Show that if $|G| = pq$ for some primes p and q, then either G is abelian or $|Z(G)| = 1$

5. Let G be a group of order 3825. Prove that if G has a normal subgroup H of order 17, then $H \subset Z(G)$.

6. If $|G| = 60$ and G has more than one 5-Sylow subgroup, then prove that G is simple.

7. If $|G| = p^e m$, where p is a prime, e and m are positive integers with $m < p$. Prove that G is not simple.

8. Prove that if $|G| = 2^e m$, where m is odd and G has a *cyclic* 2-Sylow subgroup, then G has a normal subgroup of order m.

9. Prove that the number of p-Sylow subgroups of $GL_2(\mathbb{F}_p)$ is $p + 1$

10. Let G be a group of order 315 having a *normal* 3-Sylow subgroup. Prove that $Z(G)$ contains a 3-Sylow subgroup of G. Deduce that G is abelian.

---

4. Symmetric Groups

1. Determine the disjoint cycle decomposition of the permutation (in S_n) sending $i \to n-i$ for each $i = 1, 2, \ldots, n-1$ and $n \to n$.

2. Compute the disjoint cycle decomposition of $(1\,2\,3\,4)^i$ for $i = 1$, 2, 3, and 4.

3. Prove that any proper subgroup of A_5 has order at most 12.

4. List all the conjugacy classes in S_4.

5. Find all normal subgroups in S_4.

6. Let $\sigma = (1\,2)(3\,4)$ in S_n, $n \geq 4$. Find the number of conjugates of σ. Also, determine the centralizer of σ.

7. Prove that for a prime p, the symmetric group S_p is generated by an arbitrary transposition and an arbitrary p-cycle.

8. Consider the equation $\mathcal{E} : x^p = 1$ in S_p, where p is a prime. Show that the number of solutions of $\mathcal{E}$ in S_p is $(p-1)! + 1$.

9. Find all conjugacy classes in A_5 and the number of elements in each conjugacy class.

10. If p is a prime, give explicit generators for an p-Sylow subgroup of S_{p^2}.
