NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 19, 2019

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 9 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations, and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.

- Let $n \in \mathbb{N}, n \geq 2$. The symbol $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology and $(.,.)$ will denote its usual Euclidean inner-product. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.

- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.

- The symbol i will stand for a square root of -1 in $\mathbb{C}$, the other square root being $-i$.

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.

- The space of continuous real-valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric.

- The space of continuously differentiable real-valued functions on $[a, b]$ is denoted by $\mathcal{C}^1[a, b]$ and its usual norm is the maximum of the sup-norms of the function and its derivative.

- The derivative of a function f is denoted by f' and the second and third derivatives by f'' and f''', respectively. The n-th derivative $(n > 3)$ will be denoted by $f^{(n)}$.

- The transpose (respectively, adjoint) of a (column) vector $x \in \mathbb{R}^n$ (respectively, $\mathbb{C}^n$) will be denoted by x^T (respectively, x^*). The transpose (respectively, adjoint) of any matrix A with real entries(respectively, complex entries) will be denoted by A^T (respectively, A^*).

- The symbol I will denote the identity matrix of appropriate order.

- If V is a subspace of $\mathbb{R}^N$, then
$$V^{\perp} \;=\; \{y \in \mathbb{R}^N \mid (x, y) = 0 \text{ for all } x \in V\}.$$

- The rank of any matrix, A, will be denoted by $r(A)$.

- If F is a field, $GL_n(F)$ will denote the group of invertible $n \times n$ matrices with entries from F with the group operation being matrix multiplication.

- The symbol S_n will denote the group of all permutations of n symbols $\{1, 2, \cdots, n\}$, the group operation being composition.

- The symbol $\mathbb{F}_p$ will denote the field consisting of p elements, where p is a prime.

- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 Which of the following statements are true?
a. If G is a finite group, then there exists $n \in \mathbb{N}$ such that G is isomorphic to a subgroup of $GL_n(\mathbb{R})$.
b. There exists an infinite group G such that every element, other than the identity element, is of order 2.
c. The group $GL_2(\mathbb{R})$ contains a cyclic subgroup of order 5.

1.2 Let p be a prime number. Let $n \in \mathbb{N}, n > 1$. What is the order of a p-Sylow subgroup of $GL_n(\mathbb{F}_p)$?

1.3 Give an example of a 5-Sylow subgroup of $GL_3(\mathbb{F}_5)$.

1.4 What is the number of elements of order 2 in S_4?

1.5 Let G be a group of order 10. Which of the following could be the class equation of G?
a. $10 = 1 + \cdots + 1$ (10 times).
b. $10 = 1 + 2 + 3 + 4$.
c. $10 = 1 + 1 + 1 + 2 + 5$.

1.6 Find the number of irreducible monic polynomials of degree 2 in $\mathbb{F}_p$, where p is a prime number.

1.7 Let A be an $m \times n$ matrix with real entries. Which of the following statements are true?
a. $r(A^T A) \leq r(A)$.
b. $r(A^T A) = r(A)$.
c. $r(A^T A) > r(A)$.

1.8 Let

$$V = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y - z = 0 \text{ and } x + y + t = 0\}.$$

Write down a basis for $V^\perp$.

1.9 Let $x_0 \in \mathbb{R}^3$ be the column vector such that $x_0^T = (1, 1, 1)$. Let

$$V = \{A \in \mathbb{M}_3(\mathbb{R}) \mid Ax_0 = 0\}.$$

What is the dimension of V?

1.10 Let

$$A = \begin{bmatrix} 19 & 2019 \\ 2019 & 1 \end{bmatrix}.$$

Which of the following statements are true?
a. The matrix A is diagonalizable over $\mathbb{R}$.
b. There exists a basis of $\mathbb{R}^2$ consisting of eigenvectors $\{w_1, w_2\}$ of the matrix A such that $w_1^T w_2 = 0$.
c. There exists a matrix $B \in GL_2(\mathbb{R})$ such that $B^3 = A$.

Section 2: Analysis

2.1 Let f be a function that is known to be analytic in a neighbourhood of the origin in the complex plane. Furthermore it is known that for $n \in \mathbb{N}$,

$$f^{(n)}(0) \;=\; (n-1)!(n+1)\left(\frac{n+1}{n}\right)^{(n+1)(n-1)}.$$

Find the radius of the largest circle with centre at the origin inside which the Taylor series of f defines an analytic function.

2.2 Which of the following statements are true?
a. The function $f(x) = \sin^2 x$ is uniformly continuous on $]0,\infty[$.
b. If $f :]0,\infty[\to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to 0} f(x)$ exists.
c. If $f :]0,\infty[\to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to\infty} f(x)$ exists.

2.3 Which of the following statements are true?
a. $\log x \le \frac{x}{e}$ for all $x > 0$.
b. $\log x \ge \frac{x}{e}$ for all $x > 0$.
c. $e^{\pi} > \pi^{e}$.

2.4 Let $\{r_1, \cdots, r_n, \cdots\}$ be an enumeration of the rationals in the interval $[0,1]$. Define, for $n \in \mathbb{N}$ and for each $x \in [0,1]$,

$$f_n(x) \;=\; \begin{cases} 1, & \text{if } x = r_1, \cdots, r_n, \\ 0, & \text{otherwise.} \end{cases}$$

Which of the following statements are true?
a. The function f_n is Riemann integrable over the interval $[0,1]$ for each $n \in \mathbb{N}$.
b. The sequence $\{f_n\}$ is pointwise convergent and the limit function is Riemann integrable over the interval $[0,1]$.
c. The sequence $\{f_n\}$ is pointwise convergent but the limit function is not Riemann integrable over the interval $[0,1]$.

2.5 Let $[a,b] \subset \mathbb{R}$ be a finite interval. A function $f : [a,b] \to \mathbb{R}$ is said to be of *bounded variation* if there exists a real number $L > 0$ such that for every partition $\mathcal{P}$ given by

$$\mathcal{P} \;=\; \{a = x_0 < x_1 < \cdots < x_n = b\},$$

where $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \;\le\; L.$$

Which of the following statements are true?
a. If $f : [a,b] \to \mathbb{R}$ is monotonic, then it is of bounded variation.
b. If $f \in \mathcal{C}^1[a,b]$, then it is of bounded variation.
c. If $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are of bounded variation, then $f + g$ is also of bounded variation.

2.6 Let $[a, b] \subset \mathbb{R}$ be a finite interval. Let $f : [a, b] \to \mathbb{R}$ be a bounded and Riemann integrable function. Define, for $x \in [a, b]$,

$$F(x) \;=\; \int_a^x f(t)\; dt.$$

Which of the following statements are true?
a. The function F is uniformly continuous.
b. The function F is of bounded variation.
c. The function F is differentiable on $]a, b[$.

2.7 Which of the following statements are true?
a. The sequence of functions $\{f_n\}_{n=1}^{\infty}$, defined by $f_n(x) = x^n(1-x)$, is uniformly convergent on the interval $[0, 1]$.
b. The sequence of functions $\{f_n\}_{n=1}^{\infty}$, defined by $f_n(x) = n\log(1 + \frac{x^2}{n})$, is uniformly convergent on $\mathbb{R}$.
c. The series

$$\sum_{n=1}^{\infty} 2^n \sin\frac{1}{3^n x},$$

is uniformly convergent on the interval $[1, \infty[$.

2.8 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathcal{C}^1[0, 1]$ such that $f_n(0) = 0$ for all $n \in \mathbb{N}$. Which of the following statements are true?
a. If the sequence $\{f_n\}$ converges uniformly on the interval $[0, 1]$, then the limit function is in $\mathcal{C}^1[0, 1]$.
b. If the sequence $\{f_n'\}$ is uniformly convergent over the interval $[0, 1]$, then the sequence $\{f_n\}$ is also uniformly convergent over the same interval.
c. If the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly over the interval $[0, 1]$ to a function g, then g is Riemann integrable and

$$\int_0^1 g(t)\; dt \;=\; \sum_{n=1}^{\infty} f_n(1).$$

2.9 a. Write down the Laurent series expansion of the function

$$f(z) \;=\; \frac{-1}{(z-1)(z-2)}$$

in the region $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.
b. Let the circle $\Gamma = \{z \in \mathbb{C} \mid |z| = \frac{3}{2}\}$ be described in the counter-clockwise sense. Evaluate:

$$\int_\Gamma f(z)\; dz.$$

2.10 Which of the following statements are true?
a. There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that its real part is the function e^x, where $z = x + iy$.
b. There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z$ for all z such that $|z| = 1$ and $f(z) = z^2$ for all z such that $|z| = 2$.
c. There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 1, f(4i) = i$ and for all z_j such that $1 < |z_j| < 3, j = 1, 2$, we have

$$|f(z_1) - f(z_2)| \;\leq\; |z_1 - z_2|^{\frac{\pi}{3}}.$$

Section 3: Topology

3.1 Let X be a topological space. If $A \subset X$, we denote by A°, the interior of A. Which of the following statements are true?
a. If A and B are subsets of X, then $(A \cup B)^\circ = A^\circ \cup B^\circ$.
b. If A and B are subsets of X, then $(A \cap B)^\circ = A^\circ \cap B^\circ$.
c. If $A \subset X$, then $A^\circ = (\overline{A^c})^c$, where for $B \subset X$, we denote by B^c, its complement, *i.e.* $B^c = X \backslash B$, and by $\overline{B}$, the closure of B.

3.2 Which of the following statements are true?
a. Let X be a topological space such that the set

$$\Delta = \{(x,x) \mid x \in X\}$$

is closed in $X \times X$ (with the product topology). Then X is Hausdorff.
b. Let X be a set and let Y be a Hausdorff space. Let $f : X \to Y$ be a given mapping. Define $U \subset X$ to be open in X if, and only if, $U = f^{-1}(V)$ for some set V open in Y. This defines a Hausdorff topology on X.
c. The weakest topology on $\mathbb{R}$ such that all polynomials (in a single variable) are continuous, is Hausdorff.

3.3 Which of the following statements are true?
a. The map $f : [0, 2\pi[\to S^1$, defined by $f(t) = e^{it}$, is a homeomorphism.
b. The spaces S^1 and S^2, with their topologies inherited from $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, are homeomorphic.
c. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Define $G(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y$. If $G(f)$ inherits the product topology from $X \times Y$, then X is homeomorphic to $G(f)$ via the map $x \mapsto (x, f(x))$.

3.4 Let (X, d) be a metric space. Let J be an indexing set. Consider a set of the form $S = \{x_j \in X \mid j \in J\}$ with the property that $d(x_j, x_k) = 1$ for all $j \neq k,\ j, k \in J$. Which of the following statements are true?
a. If such a set exists in X, then there exist open sets $\{U_j\}_{j \in J}$ in X such that $U_j \cap U_k = \emptyset$ for all $j \neq k,\ j, k \in J$.
b. There exists such a set S in $\mathcal{C}[0,1]$ with J being uncountable.
c. If such a set exists in X, and if X is compact, then J must be finite.

3.5 Let

$$E = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} = 1\right\},$$

with the topology inherited from $\mathbb{R}^2$. Which of the following statements are true?
a. There exists $f : E \to \mathbb{R}$ that is continuous and onto.
b. If $[\alpha, \beta]$ is any finite interval in $\mathbb{R}$, there exists $f : E \to [\alpha, \beta]$ that is continuous and onto.
c. If $[\alpha, \beta]$ is any finite interval in $\mathbb{R}$, there exists $f : E \to [\alpha, \beta]$ that is continuous, one-one and onto.

3.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous 2π- periodic function, *i.e.* for every $t \in \mathbb{R}$, we have $f(t) = f(t+2\pi)$. Which of the following statements are true?
a. There exists $t_0 \in \mathbb{R}$ such that $f(t_0) = f(-t_0)$.
b. There exists $t_0 \in \mathbb{R}$ such that $f(t_0) = f(t_0 + \frac{\pi}{2})$.
c. There exists $t_0 \in \mathbb{R}$ such that $f(t_0) = f(t_0 + \frac{\pi}{4})$.

3.7 Which of the following spaces are connected?
a. $GL_2(\mathbb{R})$, with the topology inherited from $\mathbb{M}_2(\mathbb{R})$.
b. $GL_2(\mathbb{C})$, with the topology inherited from $\mathbb{M}_2(\mathbb{C})$.
c. The space X of all symmetric matrices in $GL_2(\mathbb{R})$ with both the eigenvalues belonging to the interval $]0, 2[$, with the topology inherited from $\mathbb{M}_2(\mathbb{R})$.

3.8 Let X be a topological space. A mapping $f : X \to \mathbb{R}$ is said to be *lower semi-continuous* if the set $\{x \in X \,|f(x) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$. Which of the following statements are true?
a. If f is continuous, then it is lower semi-continuous.
b. If the set$\{x \in X \mid f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{Q}$, then f is lower semi-continuous.
c. If $\{f_n\}_{n=1}^m$ is a finite collection of lower semi-continuous functions defined on X, then $f : X \to \mathbb{R}$ defined by

$$f(x) = \min_{1 \leq n \leq m} f_n(x),$$

is lower semi-continuous.

3.9 Which of the following statements are true?
a. Let $\mathcal{F}$ be an infinite family of continuous real-valued functions on the interval $[0, 1]$ with the property that given any finite subfamily of functions $\mathcal{F}' \subset \mathcal{F}$, there exists at least one point $t \in [0, 1]$ (depending on the subfamily) such that $f(t) = 0$ for all $f \in \mathcal{F}'$. Then, there exists at least one point $t_0 \in [0, 1]$ such that $f(t_0) = 0$ for all $f \in \mathcal{F}$.
b. Let

$$\mathcal{F} = \{f \in \mathcal{C}^1[0, 1] \mid |f(t)| \leq 1 \text{ and } |f'(t)| \leq 1 \text{ for all } t \in [0, 1]\}.$$

Given any sequence in $\mathcal{F}$, there exists a subsequence which converges uniformly on $[0, 1]$.
c. If $f : [0, 1] \to \mathbb{R}$ is lower semi-continuous, then there exists $t_0 \in [0, 1]$ such that

$$f(t_0) = \min_{t \in [0,1]} f(t).$$

3.10 Let D be the closed unit disc in $\mathbb{R}^2$. Let $d(.,.)$ denote the Euclidean distance in D. Let $T : D \to D$ be a mapping such that $d(T(x), T(y)) = d(x, y)$ for all points x and y in D. Which of the following statements are true?
a. There exists $x_0 \in D$ such that $T(x_0) = x_0$.
b. The image of T is closed.
c. The mapping T is surjective.

Section 4: Calculus & Differential Equations

4.1 Evaluate:

$$\int_0^{\frac{\pi}{4}} \tan^3 x \; dx.$$

4.2 Evaluate:

$$\int\int_{\mathbb{R}^2} e^{-(19x^2+2xy+19y^2)} \; dx \; dy.$$

4.3 Evaluate:

$$\int\int_S (x^4+y^4+z^4) \; dS,$$

where $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2\}, \; a > 0.$

4.4 Evaluate $f'(3)$, where

$$f(x) \;=\; \int_{-x}^{x} \frac{1-e^{-xy}}{y} \; dy, \; x > 0.$$

4.5 Find the maximum value of $3x^2 + 2xy + y^2$ over the set $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$.

4.6 Solve: $(x+y) \; dx = (x-y) \; dy$.

4.7 Find the general solution of the system of differential equations:

$$\begin{aligned} \tfrac{dx}{dt} &= x + 2y, \\ \tfrac{dy}{dt} &= 3x + 2y. \end{aligned}$$

4.8 Find a function whose Laplace transform is given by

$$\frac{1}{s^4+s^2}.$$

4.9 Find all solutions (u, λ) where $\lambda \in \mathbb{R}$ and $u \not\equiv 0$ of the boundary value problem:

$$\begin{aligned} -u''(x) &= \lambda u(x) \text{ for } 0 < x < 1, \\ u'(0) &= u(1) \;=\; 0. \end{aligned}$$

4.10 Solve the boundary value problem:

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } B(0;R) \subset \mathbb{R}^2, \\ u &= 0 \quad \text{on } \partial B(0;R), \end{aligned}$$

where Δ is the usual Laplace operator in $\mathbb{R}^2$ and $B(0;R)$ is the open ball of radius $R(>0)$ with centre at the origin.

Section 5: Miscellaneous

5.1 Let $n \in \mathbb{N}$ be fixed. Let $C_r = \binom{n}{r}$ for $0 \le r \le n$. Evaluate:

$$C_0^2 + 3C_1^2 + \cdots + (2n+1)C_n^2.$$

5.2 Find all positive integers which leave remainders $5, 4, 3$ and 2 when divided by $6, 5, 4$ and 3, respectively.

5.3 Evaluate:

$$\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}.$$

5.4 Find the value(s) of c such that the straight line $y = 2x + c$ is tangent to the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

5.5 Find the equation of the plane passing through the point $(-1, 3, 2)$ and which is perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$.

5.6 Let $k \in \mathbb{N}$ and let $N = (k+1)^2$. Evaluate the following sum as a closed form expression in terms of k:

$$\sum_{n=1}^{N} [\sqrt{n}],$$

where $[x]$ denotes the largest integer less than, or equal to, x.

5.7 A triangle has perimeter of length 16 units and a fixed base of length 6 units. What is the maximum value of the area of the triangle?

5.8 Let B be a square of side 3 units in length. Ten points are marked in it at random. What is the probability that there exist at least two points amongst them which are of distance less than or equal to $\sqrt{2}$ units apart?

5.9 Let X be a set and let V be a (real) vector space of real-valued functions defined on X, with pointwise addition and scalar multiplication as the operations. Assume in addition that if $f \in V$, then $f^2 \in V$ and $|f| \in V$. Which of the following statements are true?
a. If f and g are in V, then $fg \in V$.
b. If f and g are in V, then $\max\{f, g\} \in V$.
c. If $f \in V$, then $f^3 \in V$.
(Note: $(fg)(x) = f(x)g(x), f^n(x) = (f(x))^n$ for $n \in \mathbb{N}$, $\max\{f, g\}(x) = \max\{f(x), g(x)\}$ and $|f|(x) = |f(x)|$ for all $x \in X$.)

5.10 A portion of a wooden cube is sawed off at each vertex so that a small equilateral triangle is formed at each corner with vertices on the edges of the cube. The 24 vertices of the new object are all connected to each other by straight lines. How many of these lines (with the exception, of course, of their end-points) lie entirely in the interior of the original cube?