NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 20, 2018

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 9 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations, and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.
- Let $n \in \mathbb{N}, n \geq 2$. The symbol $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology and $(.,.)$ will denote its usual euclidean inner-product. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.
- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.
- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
- The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric.
- The space of continuously differentiable real valued functions on $[a, b]$ is denoted by $\mathcal{C}^1[a, b]$ and its usual norm is the maximum of the sup-norms of the function and its derivative.
- The symbol ℓ_2 will denote the space of all square summable real sequences equipped with the norm

$$\|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}}, \text{ where } x = (x_k) = (x_1, x_2, \cdots, x_k, \cdots).$$

- The derivative of a function f is denoted by f' and the second and third derivatives by f'' and f''', respectively.
- The transpose (respectively, adjoint) of a (column) vector $x \in \mathbb{R}^n$ (respectively, $\mathbb{C}^n$) will be denoted by x^T (respectively, x^*). The transpose (respectively, adjoint) of a matrix $A \in \mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*).
- The symbol I will denote the identity matrix of appropriate order.
- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.
- The null space of a linear functional φ (respectively, a linear operator A) on a vector space will be denoted by $\ker(\varphi)$ (respectively, $\ker(A)$).
- If F is a field, $GL_n(F)$ will denote the group of invertible $n \times n$ matrices with entries from F with the group operation being matrix multiplication.
- The symbol S_n will denote the group of all permutations of n symbols $\{1, 2, \cdots, n\}$, the group operation being composition.
- The symbol $\mathbb{F}_p$ will denote the field consisting of p elements, where p is a prime.
- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 Find the number of elements conjugate to (1 2 3 4 5 6 7) in S_7.

1.2 What is the order of a 2-Sylow subgroup in $GL_3(\mathbb{F}_5)$?

1.3 Let H be the subgroup generated by (1 2) in S_3. Compute the normalizer, $N(H)$, of H.

1.4 Let G be a group. Which of the following statements are true?
a. The normalizer of a subgroup of G is a normal subgroup of G.
b. The centre of G is a normal subgroup of G.
c. If H is a normal subgroup of G and is of order 2, then H is contained in the centre of G.

1.5 Which of the following are prime ideals in the ring $\mathcal{C}[0,1]$?
a. $J = \{f \in \mathcal{C}[0,1] \mid f(x) = 0 \text{ for all } \frac{1}{3} \leq x \leq \frac{2}{3}\}$.

b. $J = \{f \in \mathcal{C}[0,1] \mid f(\frac{1}{3}) = f(\frac{2}{3}) = 0\}$.

c. $J = \{f \in \mathcal{C}[0,1] \mid f(\frac{1}{3}) = 0\}$.

1.6 Let

$$W = \left\{ A \in \mathbb{M}_3(\mathbb{R}) \mid A^T = -A \text{ and } \sum_{j=1}^{3} a_{1j} = 0 \right\}.$$

Write down a basis for W.

1.7 Let $A \in \mathbb{M}_5(\mathbb{C})$ be such that $(A^2 - I)^2 = 0$. Assume that A is not a diagonal matrix. Which of the following statements are true?
a. A is diagonalizable.
b. A is not diagonalizable.
c. No conclusion can be drawn about the diagonalizability of A.

1.8 Which of the following statements are true?
a. If $A \in \mathbb{M}_n(\mathbb{R})$ is such that $(Ax, x) = 0$ for all $x \in \mathbb{R}^n$, then $A = 0$.
b. If $A \in \mathbb{M}_n(\mathbb{C})$ is such that $(Ax, x) = 0$ for all $x \in \mathbb{C}^n$, then $A = 0$.
c. If $A \in \mathbb{M}_n(\mathbb{C})$ is such that $(Ax, x) \geq 0$ for all $x \in \mathbb{C}^n$, then $A = A^*$.

1.9 Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let $W = \{x \in \mathbb{R}^3 \mid Ax = 2x\}$. Construct a linear functional φ on $\mathbb{R}^3$ such that $\varphi(x_0) = 1$, where $x_0^T = (1, 2, 3)$, and $\varphi(x) = 0$ for all $x \in W$.

1.10 Let V be a finite dimensional vector space over $\mathbb{R}$ and let f and g be two non-zero linear functionals on V such that whenever $f(x) \geq 0$, we also have that $g(x) \geq 0$. Which of the following statements are true?
a. $\ker(f) \subset \ker(g)$.
b. $\ker(f) = \ker(g)$.
c. $f = \alpha g$ for some $\alpha > 0$.

Section 2: Analysis

2.1 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Which of the following statements are true?
a. If $\sum_{n=1}^{\infty} a_n < +\infty$, then $\sum_{n=1}^{\infty} a_n^5 < +\infty$.

b. If $\sum_{n=1}^{\infty} a_n^5 < +\infty$, then $\sum_{n=1}^{\infty} a_n < +\infty$.

c. If $\sum_{n=1}^{\infty} a_n^{\frac{3}{2}} < +\infty$, then $\sum_{n=1}^{\infty} \frac{a_n}{n} < +\infty$.

2.2 Which of the following functions are uniformly continuous on $\mathbb{R}$?
a. $f(x) = |\sin^3 x|$.
b. $f(x) = \tan^{-1} x$.
c. $f(x) = \sum_{n=1}^{\infty} f_n(x)$, where

$$f_n(x) = \begin{cases} n(x - n + \frac{1}{n}), & \text{if } x \in [n - \frac{1}{n}, n], \\ n(n + \frac{1}{n} - x), & \text{if } x \in [n, n + \frac{1}{n}], \\ 0, & \text{otherwise.} \end{cases}$$

2.3 Let f and g be defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational,} \end{cases} \quad \text{and } g(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Which of the following statements are true?
a. The function f is continuous almost everywhere.
b. The function f is equal to a continuous function almost everywhere.
c. The function g is equal to a continuous function almost everywhere.

2.4 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on $[0, 1]$. Assume that $f_n(x) \to f(x)$ for every $x \in [0, 1]$. Which of the following conditions imply that this convergence is uniform?
a. The function f is continuous.
b. $f_n(x) \downarrow f(x)$ for every $x \in [0, 1]$.
c. The function f is continuous and $f_n(x) \downarrow f(x)$ for every $x \in [0, 1]$.

2.5 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative continuous functions defined on $[0, 1]$. Assume that $f_n(x) \to f(x)$ for every $x \in [0, 1]$. Which of the following conditions imply that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx?$$

a. $f_n(x) \uparrow f(x)$ for every $x \in [0, 1]$.
b. $f_n(x) \leq f(x)$ for every $x \in [0, 1]$.
c. f is continuous.

2.6 Let $\{f_n\}_{n=1}^{\infty}$ and f be integrable functions on $[0,1]$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x) - f(x)|\, dx = 0.$$

Which of the following statements are true?
a. $f_n(x) \to f(x)$, as $n \to \infty$, for almost every $x \in [0,1]$.
b. $\int_0^1 f_n(x)\, dx \to \int_0^1 f(x)\, dx$, as $n \to \infty$.
c. If $\{g_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of continuous functions converging pointwise to a function g, then $\int_0^1 |f_n(x)g_n(x) - f(x)g(x)|\, dx \to 0$ as $n \to \infty$.

2.7 Let $f : [-\pi,\pi] \to \mathbb{R}$ be a continuous 2π-periodic function whose Fourier series is given by

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt).$$

Let, for each $n \in \mathbb{N}$,

$$f_n(t) \;=\; \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt),$$

and let f_0 denote the constant function $a_0/2$. Which of the following statements are true?
a. $f_n \to f$ uniformly on $[-\pi,\pi]$.
b. If $\sigma_n = (f_0 + f_1 + \cdots + f_n)/(n+1)$, then $\sigma_n \to f$ uniformly on $[-\pi,\pi]$.
c. $\int_{-\pi}^{\pi} |f_n(x) - f(x)|^2\, dx \to 0$, as $n \to \infty$.

2.8 Let $f \in \mathcal{C}^1[-\pi,\pi]$. Define, for $n \in \mathbb{N}$,

$$b_n \;=\; \int_{-\pi}^{\pi} f(t) \sin nt\, dt.$$

Which of the following statements are true?
a. $b_n \to 0$, as $n \to \infty$.
b. $nb_n \to 0$, as $n \to \infty$.
c. The series $\sum_{n=1}^{\infty} n^3 b_n^3$ is absolutely convergent.

2.9 Let Γ denote the circle in the complex plane, centred at zero and of radius 2, described in the counter-clockwise sense. Evaluate:

$$\int_{\Gamma} \frac{e^{-z}}{(z-1)^2}\, dz.$$

2.10 Which of the following statements are true?
a. There exists an entire function defined on $\mathbb{C}$ such that $f(0) = 1$ and $|f(z)| \le |z|^{-2}$ for all $z \in \mathbb{C}$ such that $|z| \ge 10$.
b. If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire function, then its image is dense in $\mathbb{C}$.
c. Let $\gamma : [0,1] \to \{z \in \mathbb{C} \mid |z| \le 1\}$ be a non-constant continuous mapping such that $\gamma(0) = 0$. Let f be an analytic function in the disc $\{z \in \mathbb{C} \mid |z| < 2\}$, such that $f(0) = 0$ and $f(1) = 1$. Then, there exists τ such that $0 < \tau < 1$ and such that for all $0 < t < \tau$, we have that $f(\gamma(t)) \ne 0$.

Section 3: Topology

3.1 Let (X, d) be a metric space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Let $x \in X$. Define a sequence $\{y_n\}_{n=1}^{\infty}$ by

$$y_{2n-1} = x_n \text{ and } y_{2n} = x, n \in \mathbb{N}.$$

Which of the following statements are true?
a. If $x_n \to x$ as $n \to \infty$, then the sequence $\{y_n\}$ is Cauchy.
b. If the sequence $\{y_n\}$ is Cauchy, then $x_n \to x$ as $n \to \infty$.
c. Let $f : X \to X$ be a mapping that maps Cauchy sequences to Cauchy sequences. Then f is continuous.

3.2 Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open and dense subsets of $\mathbb{R}^N$. Set $V = \cap_{n=1}^{\infty} V_n$. Which of the following statements are true?
a. $V \neq \emptyset$.
b. V is an open set.
c. V is dense in $\mathbb{R}^N$.

3.3 Let X be a topological space and let $U \subset X$. In which of the following cases is U open?
a. Let U be the set of invertible upper triangular matrices in $\mathbb{M}_n(\mathbb{R})$, where $n \geq 2$, and $X = \mathbb{M}_n(\mathbb{R})$.
b. Let U be the set of all 2×2 matrices with real entries such that all their eigenvalues belong to $\mathbb{C} \backslash \mathbb{R}$, and $X = \mathbb{M}_2(\mathbb{R})$.
c. Let U be the set of all complex numbers λ such that $A - \lambda I$ is invertible, where A is a given 3×3 matrix with complex entries, and $X = \mathbb{C}$.

3.4 Let X be an infinite set. Define a topology τ on X as follows:

$$\tau = \{X, \emptyset\} \cup \{U \mid X \backslash U \text{ is a non-empty finite set}\}.$$

Which of the following statements are true?
a. The topological space (X, τ) is Hausdorff.
b. The topological space (X, τ) is compact.
c. The topological space (X, τ) is connected.

3.5 Which of the following sets are connected?
a. The set of orthogonal matrices in $\mathbb{M}_n(\mathbb{R})$, where $n \geq 2$.
b. The set

$$S = \left\{ f \in \mathcal{C}[0,1] \mid \int_0^{\frac{1}{2}} f(t)\ dt - \int_{\frac{1}{2}}^1 f(t)\ dt = 1 \right\}$$

in $\mathcal{C}[0,1]$.
c. The set of all points in $\mathbb{R}^2$ with at least one coordinate being a transcendental number.

3.6 Which of the following sets are nowhere dense?
a. $S = \{A \in \mathbb{M}_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$ in $\mathbb{M}_n(\mathbb{R})$, where $n \geq 2$.
b. $S = \{x \in \ell_2 \mid x = (x_n), x_n = 0 \text{ for infinitely many } n\}$ in ℓ_2.
c. The Cantor set in $[0, 1]$.

3.7 For $x = (x_n) \in \ell_2$, define

$$T(x) = (0, x_1, x_2, \cdots) \text{ and } S(x) = (x_2, x_3, \cdots).$$

Which of the following statements are true?
a. $\|T\| = \|S\| = 1$.
b. If $A : \ell_2 \to \ell_2$ is a continuous linear operator such that $\|A - T\| < 1$, then SA is invertible.
c. If A is as above, then A is not invertible.

3.8 Let V and W be normed linear spaces and let $T : V \to W$ be a continuous linear operator. Let B be the closed unit ball in V. In which of the following cases is $\overline{T(B)}$ compact?
a. $V = \mathcal{C}^1[0,1], W = \mathcal{C}[0,1]$ and $T(f) = f$.
b. $V = W = \ell_2$ and $T(x) = (0, x_1, x_2, \cdots)$, where $x = (x_n) \in \ell_2$.
c. $V = W = \ell_2$ and $T(x) = (x_1, x_2, \cdots, x_{10}, 0, \cdots, 0, \cdots)$, where $x = (x_n) \in \ell_2$.

3.9 Which of the following statements are true?
a. The equation $x^5 + \cos^2 x = 0$ has a solution in $\mathbb{R}$.
b. The equation $2x - \cos^2 x = 0$ has a solution in $[0, 1]$.
c. The equation $x^3 - \cos^2 x = 0$ has a solution in $[-1, 0]$.

3.10 Let D denote the closed unit disc and let S^1 denote the unit circle in $\mathbb{R}^2$. Let

$$E = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \right\}.$$

Which of the following statements are true?
a. If $f : E \to E$ is continuous, than there exists $x \in E$ such that $f(x) = x$.
b. If $f : D \to S^1$ is continuous, there exists $x \in S^1$ such that $f(x) = x$.
c. If $f : S^1 \to S^1$ is continuous, there exists $x \in S^1$ such that $f(x) = x$.

Section 4: Calculus & Differential Equations

4.1 Let $A = [0,1] \times [0,1] \subset \mathbb{R}^2$. Evaluate:

$$\int\!\!\int_A \cos(\pi \max\{x,y\})\ dx\ dy.$$

4.2 Let S denote the unit sphere in $\mathbb{R}^3$. Evaluate:

$$\int_S (x^4 + y^4 + z^4)\ dS.$$

4.3 If Γ denotes the usual gamma function, evaluate $\Gamma(\frac{5}{2})$, given that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4.4 Find the general solution of the differential equation: $yy'' = (y')^2$.

4.5 A particle falling freely from rest under the influence of gravity suffers air resistance proportional to the square of its velocity at each instant. If g stands for the acceleration due to gravity and if $c > 0$ is the constant of proportionality for the air resistance, write down the initial value problem satisfied by $y(t)$, the distance travelled by the particle at time t.

4.6 In the preceding problem, if $v(t)$ is the velocity at time t, evaluate:

$$\lim_{t \to +\infty} v(t).$$

4.7 Write down the following ordinary differential equation as a system of first order differential equations:

$$y''' \ = \ y'' - x^2(y')^2.$$

4.8 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N, N \geq 2$. Let Δ denote the Laplace operator in $\mathbb{R}^N$. Consider the eigenvalue problem:

$$\begin{aligned} -\Delta u &= \lambda u, && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega. \end{aligned}$$

If $(u_i, \lambda_i), i = 1,2$ are two solutions such that $\lambda_1 \neq \lambda_2$ and $\int_\Omega |u_1(x)|^2\ dx = \int_\Omega |u_2(x)|^2\ dx = 1$, evaluate:

$$\int_\Omega u_1(x) u_2(x)\ dx.$$

4.9 Let Ω denote the unit ball in $\mathbb{R}^3$. Let Δ denote the Laplace operator in $\mathbb{R}^3$. Let u be such that

$$\begin{aligned} \Delta u &= c, && \text{in } \Omega, \\ \tfrac{\partial u}{\partial \nu} &= 1, && \text{on } \partial\Omega, \end{aligned}$$

where $\frac{\partial u}{\partial \nu}$ is the outer normal derivative of u on $\partial\Omega$. Given that c is a constant, find its value.

4.10 Let $u(x,t)$ be the solution of the following initial value problem:

$$\begin{aligned} \tfrac{\partial^2 u}{\partial t^2} &= \tfrac{\partial^2 u}{\partial x^2}, && t > 0, x \in \mathbb{R}, \\ u(x,0) &= u_0(x), && x \in \mathbb{R}, \\ \tfrac{\partial u}{\partial t}(x,0) &= 0, && x \in \mathbb{R}. \end{aligned}$$

If $u_0(x)$ vanishes outside the interval $[-1,1]$, find the interval outside which $u(\cdot,t)$ vanishes, when $t > 1$.

Section 5: Miscellaneous

5.1 Let X be a non-empty set. Let E and F be subsets of X. Define $E\Delta F = (E\backslash F) \cup (F\backslash E)$. Simplify the following expressions:

$$\text{(a)}(E\Delta F)\Delta(E \cap F), \text{ and (b) } E\Delta(E\Delta F).$$

5.2 Let X be a non-empty set and let $\{E_n\}_{n=1}^{\infty}$ be a sequence of subsets of X. Let $E \subset X$ be the set of all points in X which lie in infinitely many of the E_n. Express E in terms of the E_n using the set theoretic operations of union and intersection.

5.3 Let $n \in \mathbb{N}$ be fixed. If $0 \leq r \leq n$, let $C_r = \binom{n}{r}$. Evaluate the sum up to n terms of the series:

$$3C_1 + 7C_2 + 11C_3 + \cdots$$

5.4 Eight different dolls are to be packed in eight different boxes. If two of the boxes are too small to hold five of the dolls, in how many ways can the dolls be packed?

5.5 Given six consonants and three vowels, five-letter words are formed. What is the probability that a randomly chosen word contains three consonants and two vowels?

5.6 Find the lengths of the semi-axes of the ellipse:

$$5x^2 - 8xy + 5y^2 \ = \ 1.$$

5.7 Find the sum of the infinite series:

$$\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \cdots$$

5.8 How many zero's are there at the end of 61! ?

5.9 Which of the following statements are true?
a. The product of r consecutive positive integers is always divisible by $r!$.
b. If n is a prime number and if $0 < r < n$, then n divides $\binom{n}{r}$.
c. If $n \in \mathbb{N}$, then $n(n+1)(2n+1)$ is divisible by 6.

5.10 Let $a, b, c \in \mathbb{R}$. Find the maximum value of $ax + by + cz$ over the set

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid |x|^3 + |y|^3 + |z|^3 = 1\right\}.$$