NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 23, 2016

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 10 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.
- Let $n \in \mathbb{N}, n \geq 2$. The symbol $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.
- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.
- If X is a set and if E is a subset, the characteristic function (also called the indicator function) of E, denoted χ_E, is defined by

$$\chi_E(x) \;=\; \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
- The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric.
- The d_1-metric on a space of functions defined over a domain $X \subset \mathbb{R}$, whenever it is well-defined, is defined as follows:

$$d_1(f, g) \;=\; \int_X |f(x) - g(x)| \, dx.$$

- The derivative of a function f is denoted by f' and the second derivative by f''.
- The transpose (respectively, adjoint) of a vector $x \in \mathbb{R}^n$ (respectively, $\mathbb{C}^n$) will be denoted by x^T (respectively, x^*). The transpose (respectively, adjoint) of a matrix $A \in \mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*).
- The symbol I will denote the identity matrix of appropriate order.
- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.
- The null space of a linear functional φ (respectively, a linear operator A) on a vector space will be denoted by $\ker(\varphi)$ (respectively, $\ker(A)$).
- $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) will denote the group of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) with the group operation being matrix multiplication.
- The symbol S_n will denote the group of all permutations of n symbols $\{1, 2, \cdots, n\}$, the group operation being composition.
- The symbol $\mathbb{Z}_n$ will denote the ring of integers modulo n.
- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 With the usual notations, compute aba^{-1} in S_5 and express it as the product of disjoint cycles, where

$$a = (1\ 2\ 3)(4\ 5) \text{ and } b = (2\ 3)(1\ 4).$$

1.2 Consider the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 4 & 10 & 6 & 2 & 9 & 8 & 1 & 5 & 3 \end{pmatrix}.$$

a. Is this an odd or an even permutation?
b. What is its order in S_{10}?

1.3 Which of the following statements are true?
a. Let G be a group of order 99 and let H be a subgroup of order 11. Then H is normal in G.
b. Let H be the subgroup of S_3 consisting of the two elements $\{e, a\}$ where e is the identity and $a = (1\ 2)$. Then H is normal in S_3.
c. Let G be a finite group and let H be a subgroup of G. Define

$$W = \cap_{g \in G} gHg^{-1}.$$

Then W is a normal subgroup of G.

1.4 Consider the ring $\mathcal{C}[0, 1]$ with the operations of pointwise addition and pointwise multiplication. Give an example of an ideal in this ring which is not a maximal ideal.

1.5 Compute the (multiplicative) inverse of $4x+3$ in the field $\mathbb{Z}_{11}[x]/(x^2+1)$.

1.6 Let $A \in \mathbb{M}_5(\mathbb{R})$. If $A = (a_{ij})$, let A_{ij} denote the cofactor of the entry $a_{ij}, 1 \leq i, j \leq 5$. Let $\widehat{A}$ denote the matrix whose (ij)-th entry is $A_{ij}, 1 \leq i, j \leq 5$.
a. What is the rank of $\widehat{A}$ when the rank of A is 5?
b. What is the rank of $\widehat{A}$ when the rank of A is 3?

1.7 Write down the minimal polynomial of $A \in \mathbb{M}_n(\mathbb{R})$, where

$$A = (a_{ij}) \text{ and } a_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

1.8 Let $V = \mathbb{R}^5$ be equipped with the usual euclidean inner-product. Which of the following statements are true?
a. If W and Z are subspaces of V such that both of them are of dimension 3, then there exists $z \in Z$ such that $z \neq 0$ and $z \perp W$.
b. There exists a non-zero linear map $T : V \to V$ such that $\ker(T) \cap W \neq \{0\}$ for every subspace W of V of dimension 4.
c. Let W be a subspace of V of dimension 3. Let $T : V \to W$ be a linear map which is surjective and let $S : W \to V$ be a linear map which is injective. Then, there exists $x \in V$ such that $x \neq 0$ and such that $S \circ T(x) = 0$.

1.9 Which of the following statements are true?
a. Let $A \in \mathbb{M}_3(\mathbb{R})$ be such that $A^4 = I,\ A \neq \pm I$. Then $A^2 + I = 0$.
b. Let $A \in \mathbb{M}_2(\mathbb{R})$ be such that $A^3 = I,\ A \neq I$. Then $A^2 + A + I = 0$.
c. Let $A \in \mathbb{M}_3(\mathbb{R})$ be such that $A^3 = I,\ A \neq I$. Then $A^2 + A + I = 0$.

1.10 Find an orthogonal matrix P and a diagonal matrix D, both in $\mathbb{M}_2(\mathbb{R})$, such that $P^T AP = D$, where

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}.$$

Section 2: Analysis

2.1 Let $\{a_n\}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} \left| a_n + 3\left(\frac{n-2}{n}\right)^n \right|^{\frac{1}{n}} = \frac{3}{5}.$$

Compute $\lim_{n\to\infty} a_n$.

2.2 Let $f : [0,\infty[\to [0,\infty[$ be a continuous function such that

$$\int_0^\infty f(t)\, dt < \infty.$$

Which of the following statements are true?
a. The sequence $\{f(n)\}_{n\in\mathbb{N}}$ is bounded.
b. $f(n) \to 0$ as $n \to \infty$.
c. The series $\sum_{n=1}^\infty f(n)$ is convergent.

2.3 Let $\rho : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\rho(x) \geq 0$ for all $x \in \mathbb{R}, \rho(x) = 0$ if $|x| \geq 1$ and

$$\int_{-\infty}^\infty \rho(t)\, dt = 1.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Evaluate:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{-\infty}^\infty \rho\left(\frac{x}{\varepsilon}\right) f(x)\, dx.$$

2.4 Let $I \subset \mathbb{R}$ be an interval. A real valued function f defined on I is said to have the *intermediate value property* (IVP) if for every $a, b \in I$ such that $a < b$, the function f assumes every value between $f(a)$ and $f(b)$ in the interval (a, b). Which of the following statements are true?
a. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f has IVP.
b. If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and has IVP, then f is continuous.
c. If $f : [a,b] \to \mathbb{R}$ is a differentiable function, then f' has IVP.

2.5 Write down the Taylor expansion (about the origin) of the function

$$f(x) = \int_0^x \tan^{-1} t\, dt.$$

2.6 Use the preceding exercise to find the sum of the series:

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$$

2.7 Let $\{f_n\}$ be a sequence of continuous real valued functions defined on $\mathbb{R}$ converging uniformly on $\mathbb{R}$ to a function f. Which of the following statements are true?
a. If each of the functions f_n is bounded, then f is also bounded.
b. If each of the functions f_n is uniformly continuous, then f is also uniformly continuous.
c. If each of the functions f_n is integrable, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\ dt \ = \ \int_{-\infty}^{\infty} f(t)\ dt.$$

2.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. Consider the following statements:
A: The function f is continuous almost everywhere.
B: There exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ almost everywhere.
Which of the following implications are true?
a. A $\Rightarrow$ B.
b. B $\Rightarrow$ A.
c. A $\Leftrightarrow$ B.

2.9 Give an example of an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(P) = H$, where

$$\begin{aligned} P &= \{z \in C \mid z = x + iy, x \geq 0, y \geq 0\}, \\ H &= \{z \in C \mid z = x + iy, y \geq 0\}. \end{aligned}$$

2.10 Which of the following statements are true?
a. There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that for every $z \in \mathbb{C}, z = x + iy, \mathrm{Re} f(z) = e^x$.
b. There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that f is bounded on both the real and imaginary axes.
c. There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 1$ and for every $z \in \mathbb{C}$ such that $|z| \geq 1$, we have

$$|f(z)| \ \leq \ e^{-|z|}.$$

Section 3: Topology

3.1 Which of the following sequences $\{f_n\}$ are Cauchy?
a.
$$f_n(x) = \begin{cases} 0 & \text{if } x \notin [n-1, n+1], \\ x - n + 1 & \text{if } x \in [n-1, n], \\ n + 1 - x & \text{if } x \in [n, n+1], \end{cases}$$
in the space
$$X = \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } \int_{-\infty}^{\infty} |f(t)|\, dt < \infty \right\}$$
equipped with the d_1 metric (see, **Notation**).
b. $f_n(x) = \frac{x+n}{n}$ in the space $\mathcal{C}[0,1]$ with the usual sup-norm metric.

c. $f_n(x) = \frac{nx}{1+nx}$ in the space $\mathcal{C}[0,1]$ equipped with the usual sup-norm metric.

3.2 Let
$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x \le \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$
Let $\mathcal{C}[0,1]$ be equipped with the d_1 metric. Which of the following statements are true?
a. The sequence $\{f_n\}$ is Cauchy.
b. The sequence $\{f_n\}$ is convergent.
c. The sequence $\{f_n\}$ is not convergent.

3.3 Which of the following normed linear spaces, all equipped with the sup-norm, are complete?
a. The space of bounded uniformly continuous real valued functions defined on $\mathbb{R}$.
b. The space of continuous real valued functions defined on $\mathbb{R}$ having compact support.
c. The space of continuously differentiable real valued functions defined on $[0,1]$.

3.4 Which of the following sets, S, are dense?
a. $S = \cup_{m,n \in \mathbb{Z}} T_{m,n}$, in $\mathbb{R}^2$, where $T_{m,n}$ is the straight line passing through the origin and the point (m,n).
b. $S = GL_n(\mathbb{R})$, in $\mathbb{M}_n(\mathbb{R})$.
c. $S = \{A \in \mathbb{M}_2(\mathbb{R}) \mid \text{both eigenvalues of } A \text{ are real}\}$, in $\mathbb{M}_2(\mathbb{R})$.

3.5 Which of the following subsets of $\mathbb{R}^2$ are connected?
a. $\mathbb{R}^2 \backslash \mathbb{Q} \times \mathbb{Q}$.
b. $\{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < \infty\} \cup \{(0,0)\}$.
c. $\{(x,y) \in \mathbb{R}^2 \mid xy = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$.

3.6 Which of the following subsets are path-connected?
a. $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid y = 1\} \subset \mathbb{R}^2$.
b. $\cup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^2 \mid x = ny\} \subset \mathbb{R}^2$.
c. The set of all symmetric matrices all of whose eigenvalues are non-negative, in $\mathbb{M}_n(\mathbb{R})$.

3.7 Which of the following statements are true?
a. If $K \subset \mathbb{M}_n(\mathbb{R})$ is a compact subset, then all the eigenvalues of all the elements of K form a bounded set.
b. Let $K \subset \mathbb{M}_n(\mathbb{R})$ be defined by

$$K = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A = A^T, \mathrm{tr}(A) = 1, x^T A x \geq 0 \text{ for all } x \in \mathbb{R}^n\}.$$

Then K is compact.
c. Let $K \subset \mathcal{C}[0,1]$ (with the usual sup-norm metric) be defined by

$$K = \left\{ f \in \mathcal{C}[0,1] \mid \int_0^1 f(t)\, dt = 1 \text{ and } f(x) \geq 0 \text{ for all } x \in [0,1] \right\}.$$

Then K is compact.

3.8 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *lower semicontinuous* (lsc) if the set $f^{-1}(]-\infty, \alpha])$ is closed for every $\alpha \in \mathbb{R}$. Which of the following statements are true?
a. If $E \subset \mathbb{R}$ is a closed set, then $f = \chi_E$ (see, **Notation**) is lsc.
b. If $E \subset \mathbb{R}$ is an open set, then $f = \chi_E$ is lsc.
c. If $G = \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}$ is closed in $\mathbb{R}^2$, then f is lsc.

3.9 Let X be a non-empty compact Hausdorff space. Which of the following statements are true?
a. If X has at least n distinct points, then the dimension of $\mathcal{C}(X)$, the space of continuous real valued functions defined on X, is at least n.
b. If A and B are disjoint, non-empty and closed sets in X, there exists $f \in \mathcal{C}(X)$ such that $f(x) = -3$ for all $x \in A$ and $f(x) = 4$ for all $x \in B$.
c. If $A \subset X$ is a closed and non-empty subset and if $g : A \to \mathbb{R}$ is a continuous function, then there exists $f \in \mathcal{C}(X)$ such that $f(x) = g(x)$ for all $x \in A$.

3.10 Which of the following subsets of $\mathbb{R}^2$ are homeomorphic to the set

$$\{(x,y) \in \mathbb{R}^2 \mid xy = 1\}?$$

a. $\{(x,y) \in \mathbb{R}^2 \mid xy - 2x - y + 2 = 0\}$.
b. $\{(x,y) \in \mathbb{R}^2 \mid x^2 - 3x + 2 = 0\}$.
c. $\{(x,y) \in \mathbb{R}^2 \mid 2x^2 - 2xy + 2y^2 = 1\}$.

Section 4: Calculus and Differential Equations

4.1 Evaluate:

$$\int_0^\infty x^4 e^{-x^2}\ dx.$$

4.2 Find the arc length of the curve in the plane, whose equation in polar coordinates is given by $r = a\cos\theta$, when θ varies over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

4.3 Let $S = [0,1] \times [0,1] \subset \mathbb{R}^2$. Evaluate:

$$\int\int_S \max(x,y)\ dxdy.$$

4.4 Evaluate:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(5x^2-6xy+5y^2)}\ dxdy.$$

4.5 Let $\mathbf{x} = (x,y) \in \mathbb{R}^2$. Let $\mathbf{n}(\mathbf{x})$ denote the unit outward normal to the ellipse γ whose equation is given by

$$\frac{x^2}{4} + \frac{y^2}{9} \;=\; 1$$

at the point $\mathbf{x}$ on it. Evaluate:

$$\int_\gamma \mathbf{x}.\mathbf{n}(\mathbf{x})\ ds(\mathbf{x}).$$

4.6 Let $\omega > 0$ and let $(x_0, y_0) \in \mathbb{R}^2$. Solve:

$$\frac{dx}{dt}(t) \;=\; \omega y(t),\ \frac{dy}{dt}(t) \;=\; -\omega x(t),\ x(0) \;=\; x_0,\ y(0) \;=\; y_0.$$

4.7 Let $\omega > 0$. Compute the matrix e^A, where

$$A \;=\; \left[\begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array}\right].$$

4.8 Write down the first order system of equations equivalent to the differential equation

$$\frac{d^3y}{dx^3} \;=\; \frac{d^2y}{dx^2} - x^2\left(\frac{dy}{dx}\right)^2.$$

4.9 Consider the system of differential equations:

$$\begin{array}{rcl} x' &=& y(x^2+1) \\ y' &=& 2xy^2. \end{array}$$

a. Find the critical points of the system.
b. Find all the solution paths of the system.

4.10 Consider the boundary value problem:

$$-y''(x) \;=\; f(x) \text{ for } 0 < x < 1,\ y'(0) = y'(1) = 0.$$

In which of the following cases does there exist a solution to this problem?
a. $f(x) = \cos\pi x$.
b. $f(x) = x - \frac{1}{2}$.
c. $f(x) = \sin\pi x$.

Section 5: Miscellaneous

5.1 Write down the condition to be satisfied by the real numbers a, b, c and d in order that the sphere $x^2+y^2+z^2=1$ and the plane $ax+by+cz+d=0$ have a non-empty intersection.

5.2 In a triangle ABC, the base $AB = 6$ cms. The vertex C varies such that the area is always equal to 12 cm^2. Find the minimum value of the sum $CA+CB$.

5.3 Find the maximum value the expression $2x+3y+z$ takes as (x, y, z) varies over the sphere $x^2+y^2+z^2=1$.

5.4 Let k, r and n be positive integers such that $1 < k < r < n$. Find $\alpha_\ell, 0 \leq \ell \leq k$ such that

$$\binom{n}{r} = \sum_{\ell=0}^{k} \alpha_\ell \binom{k}{\ell}.$$

5.5 Which of the following sets are countable?
a. The set of all algebraic numbers.
b. The set of all strictly increasing infinite sequences of positive integers.
c. The set of all infinite sequences of integers which are in arithmetic progression.

5.6 Find all integer solutions of the following pair of congruences:

$$x \equiv 5 \bmod 8, \;\; x \equiv 2 \bmod 7.$$

5.7 Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \begin{cases} 1 & \text{if} s \geq \frac{1}{2}, \\ 0 & \text{if } s < \frac{1}{2}. \end{cases}$$

Evaluate:

$$\int_0^1 F(\sin \pi x)\, dx.$$

5.8 Let

$$\alpha = 1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \cdots$$

$$\beta = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots$$

$$\gamma = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \cdots$$

Which of the following numbers are rational?
a. $\frac{\alpha}{\gamma}$.
b. $\frac{\beta}{\gamma}$.
c. $\frac{\beta^2}{\gamma}$.

5.9 In how many ways can 7 people be seated around a circular table such that two particular people are always seated next to each other?

5.10 Find the sum of the following infinite series:

$$\frac{4}{20} + \frac{4.7}{20.30} + \frac{4.7.10}{20.30.40} + \cdots$$