NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 24, 2015

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 10 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.

- $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.

- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.

- The space of continuous real-valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric. The space of continuously differentiable real-valued functions on $[a, b]$ is denoted by $\mathcal{C}^1[a, b]$.

- The derivative of a function f is denoted by f' and the second derivative by f''.

- The transpose (respectively, adjoint) of a vector $x \in \mathbb{R}^n$ (respectively, $\mathbb{C}^n$) will be denoted by x^T (respectively, x^*). The transpose (respectively, adjoint) of a matrix $A \in \mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*).

- The symbol I will denote the identity matrix of appropriate order.

- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.

- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 Solve the following equation, given that its roots are in arithmetic progression:

$$x^3 - 9x^2 + 28x - 30 \;=\; 0.$$

1.2 Which of the following statements are true?
a. Every group of order 51 is cyclic.
b. Every group of order 151 is cyclic.
c. Every group of order 505 is cyclic.

1.3 Let G be the multiplicative group of non-zero complex numbers. Consider the group homomorphism $\varphi : G \to G$ given by $\varphi(z) = z^4$.
a. Identify H, the kernel of φ.
b. Identify (up to isomorphism) the quotient space G/H.

1.4 How many elements of order 7 are there in a group of order 28?

1.5 Which of the following equations can occur as the class equation of a group of order 10?
a. $10 \;=\; 1+1+1+2+5$
b. $10 \;=\; 1+2+3+4$
c. $10 \;=\; 1+1+\cdots+1$ (10 times)

1.6 Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,0,-1,1)$ and $(2,3,-1,2)$. Write down a basis for $W^\perp$, the orthogonal complement of W in $\mathbb{R}^4$ with respect to the usual euclidean inner-product.

1.7 Let B be a 5×3 matrix and let C be a 3×5 matrix, both with real entries. Set $A = BC$. What are the possible values of the rank of A when
a. both B and C have rank 3?
b. both B and C have rank 2?

1.8 Write down all the eigenvalues (along with their multiplicities) of the matrix $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ where $a_{ij} = 1$ for all $1 \le i, j \le n$.

1.9 Let $V = \mathbb{M}_n(\mathbb{C})$ be equipped with the inner-product

$$(A, B) \;=\; \mathrm{tr}(B^* A),\;\; A, B \in V.$$

Let $M \in \mathbb{M}_n(\mathbb{C})$. Define $T : V \to V$ by $T(A) = MA$. What is $T^*(A)$, where T^* denotes the adjoint of the mapping T?

1.10 Find a symmetric and positive definite matrix B such that $B^2 = A$, where

$$A \;=\; \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Section 2: Analysis

2.1 Find the radius of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{n^3 z^n}{3^n}.$$

2.2 Let (X, d) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X. Which of the following statements are true?
a. If both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, then the sequence of real numbers $\{d(x_n, y_n)\}$ is a Cauchy sequence.
b. If $d(x_n, x_{n+1}) < \frac{1}{n+1}$, then the sequence $\{x_n\}$ is a Cauchy sequence.
c. If $d(x_n, x_{n+1}) < \frac{1}{2^n}$, then the sequence $\{x_n\}$ is a Cauchy sequence.

2.3 Let $\{a_n\}$ be a sequence of real numbers such that the series $\sum_{n=1}^{\infty} |a_n|^2$ is convergent. Which of the following statements are true?
a. The series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent.
b. The series $\sum_{n=1}^{\infty} |a_n|^p$ is convergent for all $2 < p < \infty$.
c. The series $\sum_{n=1}^{\infty} |a_n|^p$ is convergent for all $1 < p < 2$.

2.4 Which of the following statements are true?
a. If $f(x) = |x|^3$ for all $x \in \mathbb{R}$, then f is twice differentiable on $\mathbb{R}$.
b. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$|f(x) - f(y)| \ \leq \ |x - y|^{\sqrt{2}}$$

for all x and y in $\mathbb{R}$, then f is a constant.
c. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and if $|f'(t)| \leq M$ for all $t \in \mathbb{R}$, then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, the function $g(x) = x + \varepsilon f(x)$ is injective.

2.5 Let $f : [0, 1] \to \mathbb{R}$ be continuous. Define

$$F(x) \ = \ \int_0^x f(t) \ dt, \ \ x \in [0, 1].$$

Which of the following statements are true?
a. The function F is Lipschitz continuous on $[0, 1]$.
b. The function F is uniformly continuous on $[0, 1]$.
c. The function F is of bounded variation on $[0, 1]$.

2.6 Let $f_n(t) = t^n$ for $n \in \mathbb{N}$. Which of the following statements are true?
a. The sequence $\{f_n\}$ converges uniformly on $[\frac{1}{4}, \frac{1}{2}]$.
b. The sequence $\{f_n\}$ converges uniformly on $[0, 1]$.
c. The sequence $\{f_n\}$ converges uniformly on $]0, 1[$.

2.7 Let $f \in \mathcal{C}^1[-\pi, \pi]$ be such that $f(-\pi) = f(\pi)$. Define

$$a_n = \int_{-\pi}^{\pi} f(t) \cos nt \; dt, \;\; n \in \mathbb{N}.$$

Which of the following statements are true?
a. The sequence $\{a_n\}$ is bounded.
b. The sequence $\{na_n\}$ converges to zero as $n \to \infty$.
c. The series $\sum_{n=1}^{\infty} n^2|a_n|^2$ is convergent.

2.8 Let Γ be a simple closed curve in the complex plane (described in the positive sense) and let z_0 be a point in the interior of this curve. Evaluate:

$$\int_\Gamma \frac{z^3 + 2z}{(z - z_0)^3} \; dz.$$

2.9 Let Γ stand for the unit circle $\{z = e^{i\theta} \; : \; -\pi \leq \theta \leq \pi\}$ in the complex plane. Let $k \in \mathbb{R}$ be a fixed constant.
a. When Γ is described in the positive sense, evaluate the integral

$$\int_\Gamma \frac{e^{kz}}{z} \; dz.$$

b. Hence, or otherwise, evaluate the integral

$$\int_0^{\pi} e^{k \cos\theta} \cos(k \sin\theta) \; d\theta.$$

2.10 Which of the following statements are true?
a. There exists a non-constant entire function which is bounded on the upper half-plane $H = \{z \in \mathbb{C} \; : \; \mathcal{I}m(z) > 0\}$.
b. There exists a non-constant entire function which takes only real values on the imaginary axis.
c. There exists a non-constant entire function which is bounded on the imaginary axis.

Section 3: Topology

3.1 Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and let $S = \{(x, y) \in \mathbb{R}^2 : ax^2 + 2hxy + by^2 = 1\}$. In which of the following cases is S homeomorphic to S^1?
a. $a = 2, h = 1, b = 2$
b. $a = 1, h = 2, b = 1$
c. $a = 1, h = 2, b = 4$

3.2 Let (X, d) be a compact metric space. Which of the following statements are true?
a. X is complete.
b. X is separable.
c. If $f : X \to \mathbb{R}$ is a continuous mapping, then it maps Cauchy sequences into Cauchy sequences.

3.3 Let J be any indexing set and let (X_j, τ_j) be toplogical spaces for each $j \in J$. Let $X = \Pi_{j\in J} X_j$ be the product space with the corresponding product topology, τ. Let $p_j : X \to X_j, j \in J$ be the coordinate projection. Which of the following statements are true?
a. The product topology τ is the weakest (*i.e.* smallest) topology on X such that each coordinate projection $p_j, j \in J$ is continuous.
b. For each $j \in J$, the mapping p_j maps open sets in X onto open sets in X_j.
c. If (X', τ') is any topological space and if $f : X' \to X$ is a given mapping, then f is continuous if, and only if, $p_j \circ f : X' \to X_j$ is continuous for each $j \in J$.

3.4 Let X be the space of all polynomials in one variable, with real coeffcients. If $p = a_0 + a_1 x + \cdots + a_n x^n \in X$, define

$$\|p\| \;=\; |a_0| + |a_1| + \cdots + |a_n|,$$

which gives the metric $d(p, q) = \|p - q\|$ on X. Which of the following statements are true?
a. The metric space X is complete.
b. Define $T : X \to X$ by

$$Tp \;=\; a_0 + a_1 x + \frac{a_2}{2} x^2 + \cdots + \frac{a_n}{n} x^n,$$

where p is as described earlier. Then T is continuous.
c. The mapping T defined above is bijective and is a homeomorphism.

3.5 State whether each of the following subsets of $\mathbb{M}_2(\mathbb{R})$ are open, closed or neither open nor closed.
a. The set of all matrices in $\mathbb{M}_2(\mathbb{R})$ such that neither eigenvalue is real.
b. The set of all matrices in $\mathbb{M}_2(\mathbb{R})$ such that both eigenvalues are real.

3.6 Let $X = \mathbb{R}^2 \backslash \{(x,y) \in \mathbb{R}^2 \ : \ 3x + 5y + 1 = 0\}$. Which of the following points lie in the same connected component of X as the origin?
a. $(-1, 2)$
b. $(2, -1)$
c. $(1, -2)$

3.7 Which of the following sets in $\mathbb{M}_n(\mathbb{R})$ are connected?
a. $S = \{A \in \mathbb{M}_n(\mathbb{R}) \ : \ A^T A = AA^T = I\}$
b. $S = \{A \in \mathbb{M}_n(\mathbb{R}) \ : \ \mathrm{tr}(A) = 1\}$
c. $S = \{A \in \mathbb{M}_n(\mathbb{R}) \ : \ x^T Ax \geq 0 \text{ for all } x \in \mathbb{R}^n\}$

3.8 Which of the following sequences $\{f_n\}$ in $\mathcal{C}[0,1]$ must contain a uniformly convergent subsequence?
a. When $|f_n(t)| \leq 3$ for all $t \in [0,1]$ and for all $n \in \mathbb{N}$.
b. When $f_n \in \mathcal{C}^1[0,1], |f_n(t)| \leq 3$ and $|f_n'(t)| \leq 5$ for all $t \in [0,1]$ and for all $n \in \mathbb{N}$.
c. When $f_n \in \mathcal{C}^1[0,1]$ and $\int_0^1 |f_n(t)| \, dt \leq 1$ for all $n \in \mathbb{N}$.

3.9 Let

$$X \ = \ \{f \in \mathcal{C}[-5,5] \ : \ f(-5) = f(5) = 0\}.$$

Which of the following statements are true?
a. There exists $f \in X$ such that $f \equiv 2$ on $[-1,0]$ and $f \equiv 3$ on $[1,2] \cup [3,4]$.
b. For every $f \in X$, there exist distinct points x_1 and x_2 in $]-5,5[$ such that $f(x_1) = f(x_2)$.
c. For every $f \in X$, there exists $x \in]-5,5[$ such that $f(x) = x$.

3.10 Let $A \in \mathbb{M}_n(\mathbb{C})$ and let

$$\rho(A) \ = \ \max\{|\lambda| \ : \ \lambda \text{ is an eigenvalue of } A\}$$

denote its spectral radius. Which of the following subsets of $\mathbb{M}_n(\mathbb{C})$ are compact?
a. $S = \{A \in \mathbb{M}_n(\mathbb{C}) \ : \ \rho(A) \leq 1\}$
b. $S = \{A \in \mathbb{M}_n(\mathbb{C}) \ : \ A = A^* \text{ and } \rho(A) \leq 1\}$
c. $S = \{A \in \mathbb{M}_n(\mathbb{C}) \ : \ AA^* = A^*A = I\}$

Section 4: Calculus and Differential Equations

4.1 Find the outward unit normal to the curve Γ, given by the equation $x^2 + 4y^2 = 4$, at a point $P = (x, y)$ lying on it.

4.2 Write down the equation of the tangent to the curve Γ given in the preceding problem at the point $P = (\sqrt{3}, \frac{1}{2})$ lying on it.

4.3 If $n = (n_1(x, y), n_2(x, y))$ is the outward unit normal at the point $P = (x, y)$ lying on the curve Γ given in Problem 4.1, evaluate

$$\int_\Gamma (n_1(x, y)x + n_2(x, y)y)\ ds.$$

4.4 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $f(x, y, z) = \varphi(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. Let B be the ball in $\mathbb{R}^3$ with centre at the origin and of radius $a > 0$. Express the integral

$$\int_B f(x, y, z)\ dxdydz$$

as an integral with respect to r.

4.5 Find the maximum area that a rectangle can have if its sides are parallel to the coordinate axes and if it is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4.6 Express the following iterated integral with the order of integration reversed:

$$\int_{-1}^{2} \int_{-x}^{2-x^2} f(x, y)\ dydx.$$

4.7 Find all the possible solutions (λ, u), where $\lambda \in \mathbb{R}$ and $u \not\equiv 0$, to the boundary value problem:

$$\begin{array}{rcl} u''(x) + \lambda u(x) & = & 0,\ x \in]0, 1[, \\ u(0) = u(1) & \text{and} & u'(0) = u'(1). \end{array}$$

4.8 Using the change of the dependent variable $z = y^{-2}$, solve the differential equation:

$$xy' + y = x^4 y^3.$$

4.9 Find the general solution of the linear system:

$$\begin{array}{rcl} x'(t) & = & 4x(t) - y(t), \\ y'(t) & = & 2x(t) + y(t). \end{array}$$

4.10 Find the extremal functions $y(x)$ of the integral:

$$\int_0^1 (y^2 - (y')^2)\ dx.$$

Section 5: Miscellaneous

5.1 Given that the equation $ax^2 + 2hxy + by^2 = 1$ represents an ellipse in the plane, what is its area?

5.2 Find the area of the circle formed by the intersection of the sphere

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 \;=\; 0$$

with the plane $x + 2y + 2z - 20 = 0$.

5.3 For a positive integer N, let $\phi(N)$ denote the number of positive integers (including unity) which are less than N and coprime to it. Which of the following statements are true?
a. If $N \neq M$, then $\phi(NM) = \phi(N)\phi(M)$.
b. If $N > 2$, then $\phi(N)$ is always even.
c. If p is a prime and if $N = p^k, k \in \mathbb{N}$, then $\phi(N) = N\left(1 - \frac{1}{p}\right)$.

5.4 Let N be a fixed positive integer and let S be the set of all positive integers (including unity) which are less than N and coprime to it. What is the sum of all the elements of S?

5.5 Which of the following statements are true?
a. For every $r \in \mathbb{N}$, there exist r consecutive composite numbers in $\mathbb{N}$.
b. For every $r \in \mathbb{N}$, the product of r consecutive numbers in $\mathbb{N}$ is always divisible by $r!$.
c. If p is a prime and if $r \in \mathbb{N}$ is such that $0 < r < p$, then p divides $\binom{p}{r}$.

5.6 Let $n \in \mathbb{N}$. Which of the following statements are true?
a. For every $n > 1$,

$$\left(\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\right)^{\frac{1}{n}} \;>\; \frac{1}{2}.$$

b. For every $n \geq 1$,

$$\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} \;<\; \frac{1}{\sqrt{2n+1}}.$$

c. For every $n > 1$,

$$1.3.5\cdots(2n-1) \;<\; n^n.$$

5.7 Let $u : \mathbb{R}^N \to \mathbb{R}$ be a given function. For $a \in \mathbb{R}$, define $a^+ = \max\{a, 0\}$. For a fixed $t \in \mathbb{R}$, set

$$v(x) \;=\; (u(x) - t)^+ + t, \;\; x \in \mathbb{R}^N.$$

Which of the following statements are true?
a. $\{x \in \mathbb{R}^N \;:\; v(x) = t\} \;=\; \{x \in \mathbb{R}^N \;:\; u(x) = t\}$.
b. $\{x \in \mathbb{R}^N \;:\; v(x) > t\} \;=\; \{x \in \mathbb{R}^N \;:\; u(x) > t\}$.
c. $\{x \in \mathbb{R}^N \;:\; v(x) > \tau\} \;=\; \{x \in \mathbb{R}^N \;:\; u(x) > \tau\}$ for all $\tau \geq t$.

5.8 Find the number of divisors of $N = 2520$ (excluding unity and N).

5.9 In how many ways can we rearrange the letters in the word

$$INDIVISIBILITY$$

such that no two 'I's are adjacent to each other?
(Note: You are allowed to express the answer in terms of binomial coefficients, factorials etc., in which case you need not explicitly calculate this number.)

5.10 BCCI has shortlisted n cricketers for a forthcoming tour. It has to select a team of r players **and** name the captain of the team. This can be done in two ways:
(Australian method) First choose the team and then select the captain from amongst the team members.
(British method) First choose the captain and then select the remaining members of the team.
Write down the combinatorial identity which expresses the fact that both methods yield the same number of outcomes.