NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 23, 2017

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions that follow

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Miscellaneous. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- **Calculators are not allowed.**

Notations

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

- The symbol i will stand for a square root of -1 in $\mathbb{C}$, while the other square root will be $-i$.

- If p is a prime number, the symbol $\mathbb{F}_p$ will denote the field of p elements, $\{0, 1, 2, \cdots, p-1\}$, with addition and multiplication modulo p.

- The symbol S_n will stand for the set of all permutations of the symbols $\{1, 2, \cdots, n\}$, which is a group under composition. If σ and τ are permutations in S_n, then $\sigma\tau$ will denote the permutation where we first apply σ and then apply τ.

- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval.

- We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).

- We denote by $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) the group (under matrix multiplication) of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and by $SL_n(\mathbb{R})$ (respectively, $SL_n(\mathbb{C})$), the subgroup of matrices with determinant equal to unity.

- The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).

- The transpose (respectively, adjoint) of a matrix A in $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*). Similarly, the transpose of a (column) vector $b \in \mathbb{R}^n$ will be the (row) vector denoted by b^T.

- The symbol I will denote the identity matrix of appropriate order.

- All logarithms, unless specified otherwise, are to the base e.

Section 1: Algebra

1.1 Let $\sigma = (14387)$ and $\tau = (548)$ be cycles in S_8. Express $\sigma\tau$ and $\tau\sigma$ (see, Notations) as the product of disjoint cycles.

1.2 Let G and G' be groups and let $\varphi : G \to G'$ be a group homomorphism which is surjective. Which of the following statements are true?
a. If G is abelian, then G' is abelian.
b. If G is cyclic, then G' is cyclic.
c. If N is a normal subgroup of G, then $\varphi(N)$ is a normal subgroup of G'.

1.3 Let $n \in \mathbb{N}, n \geq 2$. Which of the following subgroups are normal in $GL_n(\mathbb{C})$?
a. $H = \{A \in GL_n(\mathbb{C}) \mid A \text{ is upper triangular}\}$.
b. $H = \{A \in GL_n(\mathbb{C}) \mid A \text{ is diagonal}\}$.
c. $H = \{A \in GL_n(\mathbb{C}) \mid \det(A) = 1\}$.

1.4 Find the multiplicative inverse of $x + 1$ in the field $\mathbb{F}_5[x]/(x^2 + x + 1)$.

1.5 Let $N \geq 2$. Let $x \in \mathbb{R}^N, x \neq 0$, be a column vector. Find the rank of the matrix $A = xx^T$.

1.6 Let $N \geq 2$. Let $x \in \mathbb{R}^N, x \neq 0$, be a column vector. Find the condition on x such that the matrix $I - 2xx^T$ is orthogonal.

1.7 Let

$$W = \{A \in \mathbb{M}_3(\mathbb{R}) \mid A = A^T \text{ and } \mathrm{tr}(A) = 0\}.$$

Write down a basis for W.

1.8 Which of the following statements are true?
a. There exists $A \in \mathbb{M}_2(\mathbb{R})$ which is orthogonal and has 2 as an eigenvalue.
b. There exists $A \in \mathbb{M}_2(\mathbb{R})$ which is orthogonal and has i as an eigenvalue.
c. If $A \in \mathbb{M}_2(\mathbb{R})$ is orthogonal, then $\|Ax\| = \|x\|$ for every $x \in \mathbb{R}^2$, where $\|.\|$ denotes the usual euclidean norm on $\mathbb{R}^2$.

1.9 Let $A \in \mathbb{M}_2(\mathbb{R})$ be such that its eigenvalues are 1 and -1. Which of the following statements are true?
a. $A^{-1} = A$.
b. $A^{-1} = -A$.
c. No conclusion can be drawn about A^{-1}.

1.10 Solve the equation:

$$x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$$

Section 2: Analysis

2.1 Evaluate:

$$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}.$$

2.2 Evaluate:

$$\lim_{n\to\infty}\frac{1}{n}\left(\Pi_{k=1}^{n}(n+k)\right)^{\frac{1}{n}}.$$

2.3 Let $a > 0$ be a real number and let $k \in \mathbb{N}$. Evaluate:

$$\lim_{n\to\infty}\frac{\left(a+\frac{1}{n}\right)^n\left(a+\frac{2}{n}\right)^n\cdots\left(a+\frac{k}{n}\right)^n}{a^{nk}}.$$

2.4 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $a \in \mathbb{R}$. Evaluate:

$$\lim_{h\to 0}\frac{1}{h}\int_{a-h}^{a+h} f(t)\ dt.$$

2.5 Let $f_n(x) = n^2x(1-x)^n$. Evaluate:

$$\lim_{n\to\infty} f_n(x)$$

for $0 < x < 1$.

2.6 Which of the following series $\sum_{n=1}^{\infty} a_n$ are convergent?

a.

$$a_n = (n^3+1)^{\frac{1}{3}} - n.$$

b.

$$a_n = \frac{n^3+1}{2^n+1}.$$

c.

$$a_n = \frac{(n+1)(n+2)\cdots(n+n)}{n^n}.$$

2.7 Determine if each of the following series is absolutely convergent, conditionally convergent or divergent.

a.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+8}.$$

b.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(n+8)}.$$

c.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+8}.$$

2.8 Let $f(x) = e^{\frac{1}{x}}$. In which of the following domains is it uniformly continuous?
a. $]0, 1[$.
b. $]1, \infty[$.
c. $]1, 2[$.

2.9 Let $f(z) = \frac{1}{7-z}$ for $z \in \mathbb{C}, z \neq 7$. Write down the Taylor expansion of f in a neighbourhood of $z = 5$.

2.10 Let C be the contour in the complex plane consisting of two straight line segments, one from $z = 0$ to $z = i$ and the other from $z = i$ to $z = 1 + i$. Evaluate:

$$\int_C f(z)\ dz,$$

where $f(z) = y - x - 3x^2 i$, $z = x + iy$, $x, y \in \mathbb{R}$.

Section 3: Miscellaneous

3.1 Let $n \in \mathbb{N}, n \geq 2$, be a fixed positive integer. Let $C_k = \binom{n}{k}$ for $0 \leq k \leq n$. Evaluate:

$$C_1 - 2C_2 + 3C_3 - 4C_4 + \cdots + (-1)^{n-1} n C_n.$$

3.2 An algebraic number is a complex number which is the root of a polynomial with integer coefficients. Which of the following numbers are algebraic?
a. $7 + 2^{\frac{1}{3}}$.
b. $\sqrt{3} + i\sqrt{5}$.
c. $\cos \frac{2\pi}{7}$.

3.3 From a set of three women and four men, three are chosen at random to stand in a row for a photograph. What is the probability that the photograph shows a woman in the middle flanked by two men, one on either side of her?

3.4 Find the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 3$ in $\mathbb{R}^3$.

3.5 A line moves in the plane so that it passes through the point $(1, 1)$ and such that it intersects the two coordinate axes. Find the locus of the centre of the circle which passes through these two points of intersection of the line with the coordinate axes, and through the origin.

3.6 Sum the infinite series:

$$\frac{5}{3.6} \frac{1}{4^2} + \frac{5.8}{3.6.9} \frac{1}{4^3} + \frac{5.8.11}{3.6.9.12} \frac{1}{4^4} + \cdots$$

3.7 A magic square of size $N, N \geq 2$, is an $N \times N$ matrix with integer entries such that the sums of the entries of each row, each column and the two diagonals are all equal. If the entries of the magic square are made up of integers in arithmetic progression with first term a and common difference d, what is the value of this common sum?

3.8 Let $n \in \mathbb{N}$. Define

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^r, \ p \text{ a prime and } r \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Given $N \in \mathbb{N}$, compute

$$\sum_{d|N} \Lambda(d),$$

where the sum ranges over all divisors of N.

3.9 For $a \in \mathbb{R}$, define $a^+ = \max\{a, 0\}$. Let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $t \in \mathbb{R}$. Define

$$v(x) \;=\; (u(x) - t)^+ + t.$$

Which of the following statements are true?
a. $\{x \in \mathbb{R} \mid v(x) = t\} \;=\; \{x \in \mathbb{R} \mid u(x) = t\}$.
b. $\{x \in \mathbb{R} \mid v(x) > t\} \;=\; \{x \in \mathbb{R} \mid u(x) > t\}$.
c. $\{x \in \mathbb{R} \mid v(x) > \tau\} \;=\; \{x \in \mathbb{R} \mid u(x) > \tau\}$ for all $\tau \geq t$.

3.10 Let V be a real vector space of real-valued functions on a fixed set X such that (i) all constant functions are in V, and (ii) if $f \in V$, then f^2 (*i.e.* the function defined by $x \mapsto (f(x))^2$ for all $x \in X$) and $|f|$ (*i.e.* the function defined by $x \mapsto |f(x)|$ for all $x \in X$) are also in V. Which of the following statements are true?
a. If f and g belong to V, then fg (*i.e.* the function defined by $x \mapsto f(x)g(x)$ for all $x \in X$) also belongs to V.
b. If f and g belong to V, then $\max\{f, g\}$ (*i.e.* the function defined by $x \mapsto \max\{f(x), g(x)\}$ for all $x \in X$) also belongs to V.
c. If $f \in V$ and if p is a polynomial in one variable with real coefficients, then $p(f)$ (*i.e.* the function defined by $x \mapsto p(f(x))$ for all $x \in X$) also belongs to V.