NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 20, 2014

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.

- There are three parts to this test: Algebra, Analysis and Miscellaneous. Each part consists of **10** questions adding up to **30** questions in all.

- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.

- **Notations**

  - $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
  - The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
  - We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).
  - The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).
  - The derivative of a function f will be denoted by f' and the second derivative by f''.
  - All logarithms, unless specified otherwise, are to the base e.

- **Calculators are not allowed.**

Section 1: Algebra

1.1 Find the sign of the permutation σ defined below:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

1.2 Let G be an arbitrary group and let a and b be any two distinct elements of G. Which of the following statements are true?
(a) If m is the order of a and if n is the order of b, then the order of ab is the l.c.m. of m and n.
(b) The order of ab equals the order of ba.
(c) The elements ab and ba are conjugate to each other.

1.3 Let G be the group of invertible upper triangular matrices in $\mathbb{M}_2(\mathbb{R})$. If we write $A \in G$ as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix},$$

which of the following define a normal subgroup of G?
(a) $H = \{A \in G \mid a_{11} = 1\}$.
(b) $H = \{A \in G \mid a_{11} = a_{22}\}$.
(c) $H = \{A \in G \mid a_{11} = a_{22} = 1\}$.

1.4 Give an example of an ideal in the ring $\mathcal{C}[0,1]$ of all continuous real valued functions on the interval $[0,1]$ with pointwise addition and pointwise multiplication as the ring operations.

1.5 Which of the following sets of vectors form a basis for $\mathbb{R}^3$?
(a) $\{(-1,0,0),(1,1,1),(1,2,3)\}$.
(b) $\{(0,1,2),(1,1,1),(1,2,3)\}$.
(c) $\{(-1,1,0),(2,0,0),(0,1,1)\}$.

1.6 Write down a basis for the following subspace of $\mathbb{R}^4$:

$$V = \{(x,y,z,t) \in \mathbb{R}^4 \mid z = x + y,\ x + y + t = 0\}.$$

1.7 Let $A \in \mathbb{M}_2(\mathbb{R})$. Which of the following statements are true?
(a) If $(\mathrm{tr}(A))^2 > 4\det(A)$, then A is diagonalizable over $\mathbb{R}$.
(b) If $(\mathrm{tr}(A))^2 = 4\det(A)$, then A is diagonalizable over $\mathbb{R}$.
(c) If $(\mathrm{tr}(A))^2 < 4\det(A)$, then A is diagonalizable over $\mathbb{R}$.

1.8 Let V be the vector space of all polynomials in a single variable with real coefficients and of degree less than, or equal to, 3. Equip this space with the standard basis consisting of the elements $1, x, x^2$ and x^3. Consider the linear transformation $T : V \to V$ defined by

$$T(p)(x) = xp''(x) + 3xp'(x) + 2p(x), \text{ for all } p \in V.$$

Write down the corresponding matrix of T with respect to the standard basis.

1.9 With the notations and definitions of Problem 1.8 above, find $p \in V$ such that

$$xp''(x) + 3xp'(x) + 2p(x) \;=\; 11x^3 + 14x^2 + 7x + 2.$$

1.10 If α, β and γ are the roots of the equation

$$x^3 - 3x^2 + 4x - 4 \;=\; 0,$$

write down an equation of degree 3 whose roots are α^2, β^2 and γ^2.

Section 2: Analysis

2.1 Which of the following series are convergent?
(a)

$$\sum_{n=1}^{\infty} \sqrt{\frac{2n^2+3}{5n^3+1}}.$$

(b)

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{3}{2}}}.$$

(c)

$$\sum_{n=1}^{\infty} n^2 x (1-x^2)^n, \text{ where } 0 < x < 1.$$

2.2 Evaluate:

$$\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right) \right]^{\frac{1}{n}}.$$

2.3 In each of the following, evaluate the limit if the limit exists, or state that the limit does not exist if that is the case.
(a)

$$\lim_{x\to 0} \frac{[x]}{x}.$$

(b)

$$\lim_{x\to 0} x \left[\frac{1}{x}\right].$$

(c)

$$\lim_{x\to 0} \frac{\cos(\frac{\pi}{2}\cos x)}{\sin(\sin x)}.$$

(Note: The symbol $[x]$ denotes the greatest integer less than, or equal to, x.)

2.4 In each of the following find the points of continuity of the function f.
(a)

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(b)

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}. \\ x^2 - 1, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

2.5 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = a$ and let $f(a) > 0$. Evaluate:

$$\lim_{x\to a} \left(\frac{f(x)}{f(a)}\right)^{\frac{1}{\log x - \log a}}.$$

2.6 Which of the following functions are differentiable at $x = 0$?

(a)

$$f(x) = \begin{cases} \tan^{-1}\left(\frac{1}{|x|}\right), & \text{if } x \neq 0, \\ \frac{\pi}{2}, & \text{if } x = 0. \end{cases}$$

(b)

$$f(x) = |x|^{\frac{1}{2}} x.$$

(c)

$$f(x) = \begin{cases} x^2 \left|\cos \frac{\pi}{x}\right|, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

2.7 Let $x, y \in]0, \infty[$. Which of the following statements are true?
(a) $|\log(1 + x^2) - \log(1 + y^2)| \leq |x - y|$.
(b) $|\sin^2 x - \sin^2 y| \leq |x - y|$.
(c) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|$.

2.8 For what values of $x \in \mathbb{R}$ is the following function decreasing?

$$f(x) = 2x^3 - 9x^2 + 12x + 4.$$

2.9 Which of the following statements are true?
(a) If $n \in \mathbb{N}, n > 2$, then $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ is rational when n is even.
(b) If $n \in \mathbb{N}, n > 2$, then $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ is rational when n is odd.
(c) If $n \in \mathbb{N}, n > 2$, then $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$ is irrational when n is even.

2.10 A right circular cylinder is inscribed in a sphere of radius $a > 0$. What is the height of the cylinder when its volume is maximal?

Section 3: Miscellaneous

3.1 Given a function $u : \mathbb{R} \to \mathbb{R}$, define $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = -\min\{u(x), 0\}$. If u_1 and u_2 are real valued functions defined on $\mathbb{R}$, which of the following statements are true?
(a) $|u_1 - u_2| = (u_1 - u_2)^+ + (u_1 - u_2)^-$.
(b) $\max\{u_1, u_2\} = (u_1 - u_2)^+ + u_2$.
(c) $\max\{u_1, u_2\} = (u_1 - u_2)^- + u_1$.

3.2 Let $a_1, \cdots, a_n$ be positive real numbers. What is the minimum value of

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}?$$

3.3 Let n be a fixed positive integer. For $0 \le r \le n$, let C_r denote the usual binomial coefficient $\binom{n}{r}$, viz. the number of ways of choosing r objects from n objects. Evaluate:

$$\frac{C_0}{1} - \frac{C_1}{2} + \frac{C_2}{3} - \cdots + (-1)^n \frac{C_n}{n+1}.$$

3.4 Let $a, b, c \in \mathbb{R}$. Evaluate:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}.$$

3.5 Which of the following statements are true?
(a) For every $n \in \mathbb{N}$, $n^3 - n$ is divisible by 6.
(b) For every $n \in \mathbb{N}$, $n^7 - n$ is divisible by 42.
(c) Every perfect square is of the form $3m$ or $3m + 1$ for some $m \in \mathbb{N}$.

3.6 Find the sum of the following infinite series:

$$1 + \frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \cdots$$

3.7 Find the sum of the following infinite series:

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \cdots$$

3.8 Find the sum of the following infinite series:

$$\frac{1}{3!} + \frac{4}{4!} + \frac{9}{5!} + \cdots$$

3.9 Find the area of the triangle in the complex plane whose vertices are the points representing the numbers $1, \omega$ and ω^2, the cube roots of unity.

3.10 Find the equation of the plane in $\mathbb{R}^3$ which passes through the point $(-10, 5, 4)$ and which is perpendicular to the line joining the points $(4, -1, 2)$ and $(-3, 2, 3)$.