NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

October 22, 2005

Time Allowed: 150 Minutes

Maximum Marks: 60

Please read, carefully, the instructions on the following page before you write anything on this booklet

NAME:	**ROLL No.:**
Institution	

(For Official Use)

ALGEBRA	**ANALYSIS**	**GEOMETRY**	**TOTAL**

INSTRUCTIONS TO CANDIDATES

- Do not forget to write your name and roll number on the cover page.
- In the box marked 'Institution', fill in the name of the institution where you are studying the M.A./ M. Sc. course, as well as the name of the city/town and the state where it is located. This will help us to assign your centre for the interview, if you qualify for the same.
- Please ensure that your answer booklet contains 14 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **20** questions adding up to **60** questions in all.
- Answer each question, as directed, in the space provided at the end of it. Answers are to be given in the form of a word (or words, if required), a numerical value (or values) or a simple mathematical expression. **Do not write sentences.**
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), and (b)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol I will denote the identity matrix of appropriate order.

ALGEBRA

1. Let $p(x) = x^4 - 4x^3 + 2x^2 + ax + b$. Suppose that for every root λ of p, $1/\lambda$ is also a root of p. Find the values of a and b.

Answer: a = ... b = ...

2. Pick out the true statements:
(a) Every polynomial of odd degree with real coefficients has a real root.
(b) Every non-constant polynomial with real coefficients can be factorised such that every factor has real coeffcients and is of degree at most two.

Answer:

3. Let $N > 1$ be a positive integer. What is the arithmetic mean of all positive integers less than N and prime to it (including unity)?

Answer:

4. What is the order of the subgroup generated by 25(mod30) in the cyclic group $\mathbb{Z}_{30}$?

Answer:

5. Describe an isomorphism $f : \mathbb{R} \to \mathbb{R}^+$ from the additive group $\mathbb{R}$ of real numbers onto the multiplicative group $\mathbb{R}^+$ of positive real numbers.

Answer: $f(x) = \ldots\ldots$ for all $x \in \mathbb{R}$.

6. Pick out the true statements:
(a) Every group of order 36 is abelian.
(b) A group in which every element is of order at most 2 is abelian.

Answer:

7. How many abelian groups of order 8 are there (up to isomorphism)?

Answer:

8. Let G be a group and let H be a subgroup of G which has exactly two distinct cosets. Let

$$\mathcal{C} \;=\; \{H' \subset G \mid H' = gHg^{-1} \text{ for some } \; g \in G\}.$$

How many elements does the set $\mathcal{C}$ have?

Answer:

9. Let F and F' be two finite fields with nine and four elements respectively. How many field homomorphisms are there from F to F'?

Answer:

10. How many fields are there (up to isomorphism) with exactly 6 elements?

Answer:

11. Let R be a ring and let p be a polynomial of degree n with coefficients in R. Then p has at most n roots in R. - True or False?

Answer:

12. Pick out the true statements:
(a) A commutative integral domain is always a subring of a field.
(b) The ring of continuous real valued functions on the closed interval $[0, 1]$ (with pointwise addition and pointwise multiplication as the ring operations) is an integral domain.

Answer:

13. Let $\mathbb{F}_3$ be a finite field of 3 elements. How many 2×2 invertible matrices with entries from $\mathbb{F}_3$ are there?

Answer:

14. Let A be a 3×3 matrix with real entries which commutes with **all** 3×3 matrices with real entries. What is the maximum number of distinct roots that the characteristic polynomial of A can have?

Answer:

15. Let V be a vector space (over $\mathbb{R}$) of dimension 7 and let $f : V \to \mathbb{R}$ be a non-zero linear functional. Let W be a linear subspace of V such that $V = \mathrm{Ker}(f) \oplus W$ where $\mathrm{Ker}(f)$ is the null space of f. What is the dimension of W?

Answer:

16. Pick out the true statements:
(a) The eigenvalues of a unitary matrix are all equal to ± 1.
(b) The determinant of real orthogonal matrix is always ± 1.

Answer:

17. Let A be a 3×3 upper triangular matrix whose diagonal entries are $1, 2$ and -3. Express A^{-1} as a linear combination of I, A and A^2.

Answer:

18. Let $A = (a_{ij})$ be a 2×2 lower triangular matrix with diagonal entries $a_{11} = 1$ and $a_{22} = 3$. If $A^{-1} = (b_{ij})$, what are the values of b_{11} and b_{22}?

Answer: $b_{11} = \ldots\ldots\ldots\ldots$, $b_{22} = \ldots\ldots\ldots\ldots$

19. Pick out the true statements:
(a) Let A be a hermitian $N \times N$ positive definite matrix. Then, there exists a hermitian positive definite $N \times N$ matrix B such that $B^2 = A$.
(b) Let B be a nonsingular $N \times N$ matrix with real entries. Let B' be its transpose. Then $B'B$ is a symmetric and positive definite matrix.

Answer:

20. Let A be a singular $N \times N$ matrix with real entires and let $\mathbf{b}$ be a $N \times 1$ matrix (*i.e.* a column vector). Let A' and $\mathbf{b}'$ be their respective transposes. Consider the system of N linear equations in N unknowns written in matrix form as:

$$A'\mathbf{x} \;=\; \mathbf{b}.$$

Complete the following sentence: The above linear system has a solution, if, and only if, $\mathbf{b}'\mathbf{u} = 0$, for all column vectors $\mathbf{u}$ such that ...

Answer:

ANALYSIS

1. Let D_n be the open disc of radius n with centre at the point $(n,0) \in \mathbb{R}^2$. Does there exist a function $f : \mathbb{R}^2 \to \mathbb{R}$ of the form $f(x,y) = ax + by$ such that

$$\cup_{n=1}^{\infty} D_n \;=\; \{(x,y) \mid f(x,y) > 0\}?$$

If your answer is 'Yes', give the values of a and b.

Answer: No. / Yes, with $a = \ldots\ldots$, $b = \ldots\ldots$

2. Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{k^2}{k!}.$$

Answer:

3. What is the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \sqrt{\log k}\; x^k?$$

Answer:

4. What are the values of $\alpha \in \mathbb{R}$ for which the following series is convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha}.$$

Answer:

5. Evaluate:

$$\lim_{n\to\infty} (\sqrt{n^2+n} - \sqrt{n^2+1}).$$

Answer:

6. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}}.$$

Answer:

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. List those amongst the following properties which will ensure that f is uniformly continuous:
(a) for all x and $y \in \mathbb{R}$,

$$|f(x) - f(y)| \ \leq \ |x - y|^{\frac{1}{2}}.$$

(b)

$$f(x) = \sum_{n=1}^{\infty} \frac{g(x-n)}{2^n}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function such that the series converges for each $x \in \mathbb{R}$.

Answer:

8. Let $f(x) = [x] + (x - [x])^2$ for $x \in \mathbb{R}$, where $[x]$ denotes the largest integer not exceeding x. What is the set of all values taken by the function f?

Answer:

9. Let n be a positive integer. Let $f(x) = x^{n+2} \sin \frac{1}{x}$ if $x \neq 0$ and let $f(0) = 0$. For what value of n will f be twice differentiable but with its second derivative discontinuous at $x = 0$?

Answer:

10. What is the coefficient of x^8 in the expansion of $x^2 \cos(x^2)$ around $x = 0$?

Answer:

11. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $|g'(x)| \leq M$ for all $x \in \mathbb{R}$. For what values of ε will the function $f(x) = x + \varepsilon g(x)$ be necessarily one-to-one?

Answer:

12. Differentiate:

$$f(x) = \int_{x}^{x^2} e^{t^2}\,dt, \;\; x > 1.$$

Answer: $f'(x) \;=$

13. Let

$$f(x) = \cos x - 1 + \frac{x^2}{2!}, \;\; x \in \mathbb{R}.$$

Pick out the true statements:
(a) $f(x) \geq 0$ for $x \geq 0$ and $f(x) \leq 0$ for $x \leq 0$.
(b) f is a decreasing function on the entire real line.

Answer:

14. Let $f : [-1, 2] \to \mathbb{R}$ be given by $f(x) = 2x^3 - x^4 - 10$. What is the value of x where f assumes its minimum value?

Answer:

15. Does the integral $\int_0^1 \log x \; dx$ exist?. If it exists, give its value.

Answer:No. / Yes, the value is ......

16. Let

$$f(x, y) \;=\; \begin{cases} 1 & \text{if } \; |y| \geq x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Does the directional derivative of f in the direction of $(1, 10^{-6})$ at the point $(0, 0)$ exist? If yes, give its value.

Answer:No. / Yes, and the value is ......

17. Write down a solution, other than the zero function, of the differential equation:

$$yy'' - (y')^2 = 0.$$

Answer:

18. Write down the general solution of the system of differential equations:

$$f'(x) \ = \ g(x); \ g'(x) \ = \ -f(x).$$

Answer:

19. Let u and v be two twice continuously differentiable functions on $\mathbb{R}^2$ satisfying the Cauchy-Riemann equations. Let $z = x + iy$. Define $f(z) = u(x,y) + iv(x,y)$. Express the complex derivative of f, *i.e.* $f'(z)$, in terms of the partial derivatives of u and v.

Answer:

20. Let $\{X, d\}$ be a metric space and let $E \subset X$. For $x \in X$, define

$$d(x, E) \ = \ \inf_{y \in E} d(x, y).$$

Pick out the true statements:
(a) $|d(x, E) - d(y, E)| \leq d(x, y)$ for all x and $y \in X$.
(b) $d(x, E) = d(x, y_0)$ for some $y_0 \in E$.

Answer:

GEOMETRY

1. What are the equations of the circles of radius $\sqrt{17}$, with centres on the x-axis and passing through the point $(0,1)$?

Answer:

2. A point moves so that the sum of its distances from two fixed points is a constant. What is the path traced by this point?

Answer:

3. A point moves so that the line segments joining it to the points $(2,0)$ and $(0,2)$ are always perpendicular. What is the equation of its path?

Answer:

4. How many points are there on the curve $x^2 - 4y^2 = 1$ at which the tangents are parallel to the line $x - 2y = 0$?

Answer:

5. A non-empty set in the Euclidean plane is said to be convex if the line segments joining any two points in the set is completely contained in the set. Pick out the convex sets:
(a) The intersection of two circular discs of radius 2 having centres $(0,0)$ and $(3,0)$.
(b) The set of all $(x,y) \in \mathbb{R}^2$ satisfying the inequalities:

$$\begin{aligned} x + y &\geq 1, \\ x + 3y &\geq 2, \\ x &\leq 2. \end{aligned}$$

Answer:

6. Let $ABCD$ be a parallellogram in the plane with area 1 and such that B and C lie on the x-axis and A and D lie on the line $y = h$. With BC as base, triangles BCE of area 1/4 are constructed. Write down the equation of the locus of E.

Answer:

7. What is the intersection of the planes $x+2y+3z+1 = 0,\ x-y+z-1 = 0$ and $y + z = 0$? Is it
(a) a point, or
(b) the empty set?

Answer:

8. Do the lines $x - 1 = y - 2 = z - 3$ and $2(x + 1) = 3(y - 1) = 6(z - 4)$ intersect? If yes, give the point of intersection.

Answer: No./ Yes, the point of intersection is ......

9. Let $ax^2 + 2hxy + by^2 = 1$ represent an ellipse and let λ_1 and λ_2, where $\lambda_1 > \lambda_2 > 0$, be the eigenvalues of the matrix

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix}.$$

Express the lengths of the semi-axes of the ellipse in terms of λ_1 and λ_2.

Answer: Semi-major axis = ..., semi-minor axis = ...

10. The normal to the sphere with centre at $(1, 1, 0)$ and radius 5 at the point $(1, 4, 4)$ is perpendicular to the plane $x + 3y - 4z = 2$. True or False?

Answer:

11. What are the points of intersection of the line $y - x = 1$ and the set represented by the equation $xy(2x^2 + 2y^2 - 5) = 0$?

Answer:

12. Pick out the bounded sets from amongst those represented by the following equations:
(a) $xy = 1$.
(b) $2x^2 + xy - y^2 = 0$.

Answer:

13. What do the following equations represent in three-dimensional Euclidean space?
(a) $x^2 - z^2 = 0$.
(b) $x^2 + y^2 - z^2 = 0$.

Answer: (a)
(b)

14. Find the centre and the eccentricity of the hyperbola:

$$2(y-1)^2 - 2(x-3)^2 = 9.$$

Answer:

15. Assume that the parabola $y^2 = 8x$ is the cross section of a parabolic reflector. A ray of light from a source at the point $(2, 0)$ strikes the point $(2, 4)$. Through which of the following points does the reflected ray pass?
(a) $(1, 4)$
(b) $(5, 4)$

Answer:

16. What figure does the equation $r^2 \cos^2(\theta - \pi/3) = 2$ (in polar coordinates) represent?

Answer:

17. A particle moves from the point $(0, 0)$ to the point $(1, 1)$ in a straight line and then moves along the arc of the circle $x^2 + y^2 = 2$ for a distance of $\pi/2\sqrt{2}$ and then moves back to the origin in a straight line. What is the area enclosed by its trajectory?

Answer:

18. What is the shape of the region whose boundary is given by

$$(2x^2 + xy - y^2 + 6y - 8)(x - y) = 0?$$

Answer:

19. Find the area of the region enclosed by the intersection of the surfaces: $x^2 + y^2 + z^2 - 2x - 2y - 2z = 1$ and $x + y + z = 1$.

Answer:

20. How many common tangents can be drawn to the curves $2x^2 + 4y^2 = 1$ and $x^2 + y^2 - 4x - 4y + 7 = 0$?

Answer: