Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

Special Instructions / Useful Data

$\mathbb{R}$	The set of all real numbers
P^T	Transpose of the matrix P
$\mathbb{R}^n$	$\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R}, i = 1, 2, \dots, n \right\}$
f'	Derivative of the differentiable function f
I_n	$n \times n$ identity matrix
$P(E)$	Probability of the event E
$E(X)$	Expectation of the random variable X
$Var(X)$	Variance of the random variable X
i.i.d.	Independently and identically distributed
$U(a, b)$	Continuous uniform distribution on $(a, b), -\infty < a < b < \infty$
$Exp(\lambda)$	Exponential distribution with probability density function, for $\lambda > 0$, $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$\Phi(a)$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{u^2}{2}} du$
χ_n^2	Central Chi-squared distribution with n degrees of freedom
$t_{n,\alpha}$	A constant such that $P\left(X > t_{n,\alpha}\right) = \alpha$, where X has Student's t-distribution with n degrees of freedom
$n!$	$n(n-1) \cdots 3 \cdot 2 \cdot 1$ for $n = 1, 2, 3 \dots$, and $0! = 1$

$$\Phi(1.65) = 0.950, \quad \Phi(1.96) = 0.975$$
$$t_{4,\,0.05} = 2.132, \qquad t_{4,\,0.10} = 1.533$$

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 Let $\{x_n\}_{n\geq 1}$ be a sequence of positive real numbers. Which one of the following statements is always TRUE?

(A) If $\{x_n\}_{n\geq 1}$ is a convergent sequence, then $\{x_n\}_{n\geq 1}$ is monotone

(B) If $\{x_n^2\}_{n\geq 1}$ is a convergent sequence, then the sequence $\{x_n\}_{n\geq 1}$ does not converge

(C) If the sequence $\{|x_{n+1} - x_n|\}_{n\geq 1}$ converges to 0, then the series $\sum_{n=1}^{\infty} x_n$ is convergent

(D) If $\{x_n\}_{n\geq 1}$ is a convergent sequence, then $\{e^{x_n}\}_{n\geq 1}$ is also a convergent sequence

Q.2 Consider the function $f(x, y) = x^3 - 3xy^2,\ x, y \in \mathbb{R}$. Which one of the following statements is TRUE?

(A) f has a local minimum at $(0, 0)$

(B) f has a local maximum at $(0, 0)$

(C) f has global maximum at $(0, 0)$

(D) f has a saddle point at $(0, 0)$

Q.3 If $F(x) = \int_{x^3}^{4} \sqrt{4 + t^2}\, dt$, for $x \in \mathbb{R}$, then $F'(1)$ equals

(A) $-3\sqrt{5}$ (B) $-2\sqrt{5}$ (C) $2\sqrt{5}$ (D) $3\sqrt{5}$

Q.4 Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$. Suppose that $\begin{bmatrix}3\\-2\end{bmatrix} = \alpha\begin{bmatrix}1\\2\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}3\\-2\end{bmatrix}\right) = \begin{bmatrix}a\\b\end{bmatrix}$. Then $\alpha + \beta + a + b$ equals

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$ (C) $\frac{5}{3}$ (D) $\frac{7}{3}$

Q.5 Two biased coins C_1 and C_2 have probabilities of getting heads $\frac{2}{3}$ and $\frac{3}{4}$, respectively, when tossed. If both coins are tossed independently two times each, then the probability of getting exactly two heads out of these four tosses is

(A) $\frac{1}{4}$ (B) $\frac{37}{144}$ (C) $\frac{41}{144}$ (D) $\frac{49}{144}$

Q.6 Let X be a discrete random variable with the probability mass function

$$P(X=n)=\begin{cases}\frac{-2c}{n}, & n=-1,-2,\\ d, & n=0,\\ c\,n, & n=1,2,\\ 0, & \text{otherwise,}\end{cases}$$

where c and d are positive real numbers. If $P(|X|\le 1)=3/4$, then $E(X)$ equals

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

Q.7 Let X be a Poisson random variable and $P(X=1)+2\,P(X=0)=12\,P(X=2)$. Which one of the following statements is TRUE?

(A) $0.40 < P(X=0) \le 0.45$ (B) $0.45 < P(X=0) \le 0.50$

(C) $0.50 < P(X=0) \le 0.55$ (D) $0.55 < P(X=0) \le 0.60$

Q.8 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. discrete random variables with the probability mass function

$$P(X_1=m)=\begin{cases}\frac{(\log_e 2)^m}{2(m!)}, & m=0,1,2,\ldots,\\ 0, & \text{otherwise.}\end{cases}$$

If $S_n = X_1+X_2+\cdots+X_n$, then which one of the following sequences of random variables converges to 0 in probability?

(A) $\frac{S_n}{n\log_e 2}$ (B) $\frac{S_n-n\log_e 2}{n}$ (C) $\frac{S_n-\log_e 2}{n}$ (D) $\frac{S_n-n}{\log_e 2}$

Q.9 Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with the probability density function

$$f(x)=\frac{1}{2\sqrt{2\pi}}\left[e^{-\frac{1}{2}(x-2\mu)^2}+e^{-\frac{1}{2}(x-4\mu)^2}\right],\quad -\infty<x<\infty.$$

If $T = X_1+X_2+\cdots+X_n$, then which one of the following is an unbiased estimator of μ?

(A) $\frac{T}{n}$ (B) $\frac{T}{2n}$ (C) $\frac{T}{3n}$ (D) $\frac{T}{4n}$

Q.10 Let $X_1, X_2, \dots, X_n$ be a random sample from a $N(\theta, 1)$ distribution. Instead of observing $X_1, X_2, \dots, X_n$, we observe $Y_1, Y_2, \dots, Y_n$, where $Y_i = e^{X_i}$, $i = 1, 2, \dots, n$. To test the hypothesis

$$H_0: \theta = 1 \text{ against } H_1: \theta \neq 1$$

based on the random sample $Y_1, Y_2, \dots, Y_n$, the rejection region of the likelihood ratio test is of the form, for some $c_1 < c_2$,

(A) $\sum_{i=1}^{n} Y_i \leq c_1$ or $\sum_{i=1}^{n} Y_i \geq c_2$

(B) $c_1 \leq \sum_{i=1}^{n} Y_i \leq c_2$

(C) $c_1 \leq \sum_{i=1}^{n} \log_e Y_i \leq c_2$

(D) $\sum_{i=1}^{n} \log_e Y_i \leq c_1$ or $\sum_{i=1}^{n} \log_e Y_i \geq c_2$

Q. 11 – Q. 30 carry two marks each.

Q.11 $\sum_{n=4}^{\infty} \frac{6}{n^2-4n+3}$ equals

(A) $\frac{5}{2}$ (B) 3 (C) $\frac{7}{2}$ (D) $\frac{9}{2}$

Q.12 $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(\pi^n+e^n)^{1/n} \log_e n}$ equals

(A) $\frac{1}{\pi}$ (B) $\frac{1}{e}$ (C) $\frac{e}{\pi}$ (D) $\frac{\pi}{e}$

Q.13 A possible value of $b \in \mathbb{R}$ for which the equation $x^4 + bx^3 + 1 = 0$ has no real root is

(A) $\frac{-11}{5}$ (B) $\frac{-3}{2}$ (C) 2 (D) $\frac{5}{2}$

Q.14 Let the Taylor polynomial of degree 20 for $\frac{1}{(1-x)^3}$ at $x = 0$ be $\sum_{n=0}^{20} a_n x^n$. Then a_{15} is

(A) 136 (B) 120 (C) 60 (D) 272

Q.15 The length of the curve $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 7$ from $x = 1$ to $x = 8$ equals

(A) $\frac{99}{8}$ (B) $\frac{117}{8}$ (C) $\frac{99}{4}$ (D) $\frac{117}{4}$

Q.16 The volume of the solid generated by revolving the region bounded by the parabola $x = 2y^2 + 4$ and the line $x = 6$ about the line $x = 6$ is

(A) $\frac{78\pi}{15}$ (B) $\frac{91\pi}{15}$ (C) $\frac{64\pi}{15}$ (D) $\frac{117\pi}{15}$

Q.17 Let P be a 3×3 non-null real matrix. If there exist a 3×2 real matrix Q and a 2×3 real matrix R such that $P = QR$, then

(A) $P\boldsymbol{x} = \mathbf{0}$ has a unique solution, where $\mathbf{0} \in \mathbb{R}^3$
(B) there exists $\boldsymbol{b} \in \mathbb{R}^3$ such that $P\boldsymbol{x} = \boldsymbol{b}$ has no solution
(C) there exists a non-zero $\boldsymbol{b} \in \mathbb{R}^3$ such that $P\boldsymbol{x} = \boldsymbol{b}$ has a unique solution
(D) there exists a non-zero $\boldsymbol{b} \in \mathbb{R}^3$ such that $P^T\boldsymbol{x} = \boldsymbol{b}$ has a unique solution

Q.18 If $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$ and $6P^{-1} = aI_3 + bP - P^2$, then the ordered pair (a, b) is

(A) $(3, 2)$ (B) $(2, 3)$ (C) $(4, 5)$ (D) $(5, 4)$

Q.19 Let E, F and G be any three events with $P(E) = 0.3$, $P(F|E) = 0.2$, $P(G|E) = 0.1$ and $P(F \cap G|E) = 0.05$. Then $P(E - (F \cup G))$ equals

(A) 0.155 (B) 0.175 (C) 0.225 (D) 0.255

Q.20 Let E and F be any two independent events with $0 < P(E) < 1$ and $0 < P(F) < 1$. Which one of the following statements is **NOT** TRUE?

(A) $P(\text{Neither } E \text{ nor } F \text{ occurs}) = (P(E) - 1)(P(F) - 1)$
(B) $P(\text{Exactly one of } E \text{ and } F \text{ occurs}) = P(E) + P(F) - P(E)P(F)$
(C) $P(E \text{ occurs but } F \text{ does not occur}) = P(E) - P(E \cap F)$
(D) $P(E \text{ occurs given that } F \text{ does not occur}) = P(E)$

Q.21 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{1}{3} x^7 e^{-x^2}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the distribution of the random variable $W = 2X^2$ is

(A) χ_2^2 (B) χ_4^2 (C) χ_6^2 (D) χ_8^2

Q.22 Let X be a continuous random variable with the probability density function

$$f(x) = \frac{e^x}{(1+e^x)^2}, \quad -\infty < x < \infty.$$

Then $E(X)$ and $P(X > 1)$, respectively, are

(A) 1 and $(1+e)^{-1}$
(B) 0 and $2(1+e)^{-2}$
(C) 2 and $(2+2e)^{-1}$
(D) 0 and $(1+e)^{-1}$

Q.23 The lifetime (in years) of bulbs is distributed as an $Exp(1)$ random variable. Using Poisson approximation to the binomial distribution, the probability (round off to 2 decimal places) that out of the fifty randomly chosen bulbs at most one fails within one month equals

(A) 0.05 (B) 0.07 (C) 0.09 (D) 0.11

Q.24 Let X follow a beta distribution with parameters $m\ (> 0)$ and 2. If $P\left(X \leq \frac{1}{2}\right) = \frac{1}{2}$, then $Var(X)$ equals

(A) $\frac{1}{10}$ (B) $\frac{1}{20}$ (C) $\frac{1}{25}$ (D) $\frac{1}{40}$

Q.25 Let X_1, X_2 and X_3 be i.i.d. $U(0,1)$ random variables. Then $P(X_1 > X_2 + X_3)$ equals

(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

Q.26 Let X and Y be i.i.d. $U(0,1)$ random variables. Then $E(X|X > Y)$ equals

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Q.27 Let -1 and 1 be the observed values of a random sample of size two from $N(\theta, \theta)$ distribution. The maximum likelihood estimate of θ is

(A) 0 (B) 2 (C) $\frac{-\sqrt{5}-1}{2}$ (D) $\frac{\sqrt{5}-1}{2}$

Q.28 Let X_1 and X_2 be a random sample from a continuous distribution with the probability density function

$$f(x) = \begin{cases} \frac{1}{\theta}\, e^{-\frac{x-\theta}{\theta}}, & x > \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. If $X_{(1)} = \min\{X_1, X_2\}$ and $\overline{X} = \frac{(X_1+X_2)}{2}$, then which one of the following statements is TRUE?

(A) $\left(\overline{X}, X_{(1)}\right)$ is sufficient and complete

(B) $\left(\overline{X}, X_{(1)}\right)$ is sufficient but not complete

(C) $\left(\overline{X}, X_{(1)}\right)$ is complete but not sufficient

(D) $\left(\overline{X}, X_{(1)}\right)$ is neither sufficient nor complete

Q.29 Let $X_1, X_2, \dots, X_n$ be a random sample from a continuous distribution with the probability density function $f(x)$. To test the hypothesis

$H_0: f(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2}, -\infty < x < \infty$ against $H_1: f(x) = e^{-2|x|}, -\infty < x < \infty$,

the rejection region of the most powerful size α test is of the form, for some $c > 0$,

(A) $\sum_{i=1}^{n}(X_i - 1)^2 \geq c$

(B) $\sum_{i=1}^{n}(X_i - 1)^2 \leq c$

(C) $\sum_{i=1}^{n}(|X_i| - 1)^2 \geq c$

(D) $\sum_{i=1}^{n}(|X_i| - 1)^2 \leq c$

Q.30 Let $X_1, X_2, \dots, X_n$ be a random sample from a $N(\theta, 1)$ distribution. To test $H_0: \theta = 0$ against $H_1: \theta = 1$, assume that the critical region is given by $\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{3}{4}$. Then the minimum sample size required so that $P(\text{Type I error}) \leq 0.05$ is

(A) 3

(B) 4

(C) 5

(D) 6

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 Let $\{x_n\}_{n\geq 1}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} x_n$ converges. Which of the following statements is (are) always TRUE?

(A) The series $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$ converges

(B) $\lim_{n\to\infty} n\, x_n = 0$

(C) The series $\sum_{n=1}^{\infty} \sin^2 x_n$ converges

(D) The series $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{1+\sqrt{x_n}}$ converges

Q.32 Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$. Which of the following statements is (are) always TRUE?

(A) If f is differentiable at 0 and $f'(0) = 0$, then f has a local maximum or a local minimum at 0

(B) If f has a local minimum at 0, then f is differentiable at 0 and $f'(0) = 0$

(C) If $f'(x) < 0$ for all $x < 0$ and $f'(x) > 0$ for all $x > 0$, then f has a global maximum at 0

(D) If $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, then f has a global maximum at 0

Q.33 Let P be a 2×2 real matrix such that every non-zero vector in $\mathbb{R}^2$ is an eigenvector of P. Suppose that λ_1 and λ_2 denote the eigenvalues of P and $P\begin{bmatrix}\sqrt{2}\\ \sqrt{3}\end{bmatrix} = \begin{bmatrix}2\\ t\end{bmatrix}$ for some $t \in \mathbb{R}$. Which of the following statements is (are) TRUE?

(A) $\lambda_1 \neq \lambda_2$

(B) $\lambda_1 \lambda_2 = 2$

(C) $\sqrt{2}$ is an eigenvalue of P

(D) $\sqrt{3}$ is an eigenvalue of P

Q.34 Let P be an $n \times n$ non-null real skew-symmetric matrix, where n is even. Which of the following statements is (are) always TRUE?

(A) $P\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions, where $\boldsymbol{0} \in \mathbb{R}^{\boldsymbol{n}}$

(B) $P\boldsymbol{x} = \lambda\boldsymbol{x}$ has a unique solution for every non-zero $\lambda \in \mathbb{R}$

(C) If $Q = (I_n + P)(I_n - P)^{-1}$, then $Q^T Q = I_n$

(D) The sum of all the eigenvalues of P is zero

Q.35 Let X be a random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \dfrac{1+x^2}{10}, & 0 \leq x < 1, \\ \dfrac{3+x^2}{10}, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

Which of the following statements is (are) TRUE?

(A) $P(1 < X < 2) = \frac{3}{10}$

(B) $P(1 < X \leq 2) = \frac{3}{5}$

(C) $P(1 \leq X < 2) = \frac{1}{2}$

(D) $P(1 \leq X \leq 2) = \frac{4}{5}$

Q.36 Let X and Y be i.i.d. $Exp(\lambda)$ random variables. If $Z = \max\{X - Y, 0\}$, then which of the following statements is (are) TRUE?

(A) $P(Z = 0) = \frac{1}{2}$

(B) The cumulative distribution function of Z is $F(z) = \begin{cases} 0, & z < 0, \\ 1 - \frac{1}{2}e^{-\lambda z}, & z \geq 0 \end{cases}$

(C) $P(Z = 0) = 0$

(D) The cumulative distribution function of Z is $F(z) = \begin{cases} 0, & z < 0, \\ 1 - e^{-\lambda z/2}, & z \geq 0 \end{cases}$

Q.37 Let the discrete random variables X and Y have the joint probability mass function

$$P(X = m, Y = n) = \begin{cases} \dfrac{e^{-2}}{m!\,n!}, & m = 0,1,2,\dots;\ n = 0,1,2,\dots, \\ 0, & \text{otherwise.} \end{cases}$$

Which of the following statements is (are) TRUE?

(A) The marginal distribution of X is Poisson with mean 2

(B) The random variables X and Y are independent

(C) The covariance between X and $X + \sqrt{3}\,Y$ is 1

(D) $P(Y = n) = (n+1)P(Y = n+1)$ for $n = 0,1,2,\dots$

Q.38 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. continuous random variables with the probability density function

$$f(x) = \begin{cases} 2e^{-2\left(x-\frac{1}{2}\right)}, & x \geq \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

If $S_n = X_1 + X_2 + \cdots + X_n$ and $\overline{X}_n = S_n/n$, then the distributions of which of the following sequences of random variables converge(s) to a normal distribution with mean 0 and a finite variance?

(A) $\frac{S_n - n}{\sqrt{n}}$ (B) $\frac{S_n}{\sqrt{n}}$ (C) $\sqrt{n}\left(\overline{X}_n - 1\right)$ (D) $\frac{\sqrt{n}\left(\overline{X}_n - 1\right)}{2}$

Q.39 Let $X_1, X_2, \ldots, X_n$ be a random sample from a $U(\theta,\ 0)$ distribution, where $\theta < 0$. If $T_n = \min\{X_1, X_2, \ldots, X_n\}$, then which of the following sequences of estimators is (are) consistent for θ?

(A) T_n (B) $T_n - 1$ (C) $T_n + \frac{1}{n}$ (D) $T_n - 1 - \frac{1}{n^2}$

Q.40 Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with the probability density function, for $\lambda > 0$,

$$f(x) = \begin{cases} 2\lambda x e^{-\lambda x^2}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

To test the hypothesis $H_0: \lambda = \frac{1}{2}$ against $H_1: \lambda = \frac{3}{4}$ at the level $\alpha\ (0 < \alpha < 1)$, which of the following statements is (are) TRUE?

(A) The most powerful test exists for each value of α

(B) The most powerful test does not exist for some values of α

(C) If the most powerful test exists, it is of the form: Reject H_0 if $X_1^2 + X_2^2 + \cdots + X_n^2 \leq c$ for some $c > 0$

(D) If the most powerful test exists, it is of the form: Reject H_0 if $X_1^2 + X_2^2 + \cdots + X_n^2 \geq c$ for some $c > 0$

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 $\lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{n+n}}{n\sqrt{n}}$ (round off to 2 decimal places) equals ________

Q.42 Let $f: [0,2] \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq |x - y|^{4/3}$ for all $x, y \in [0,2]$. If $\int_0^2 f(x)dx = \frac{2}{3}$, then $\sum_{k=1}^{2019} f\left(\frac{1}{k}\right)$ equals ________

Q.43 The value (round off to 2 decimal places) of the double integral

$$\int_0^9 \int_{\sqrt{x}}^3 \frac{1}{1+y^3}\, dy dx$$

equals ________

Q.44 If $\begin{bmatrix} \frac{\sqrt{5}}{3} & -\frac{2}{3} & c \\ \frac{2}{3} & \frac{\sqrt{5}}{3} & d \\ a & b & 1 \end{bmatrix}$ is a real orthogonal matrix, then $a^2 + b^2 + c^2 + d^2$ equals ________

Q.45 Two fair dice are tossed independently and it is found that one face is odd and the other one is even. Then the probability (round off to 2 decimal places) that the sum is less than 6 equals ________

Q.46 Let X be a random variable with the moment generating function

$$M_X(t) = \left(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}\right)^2, \qquad -\infty < t < \infty.$$

Using Chebyshev's inequality, the upper bound for $P\left(|X| > \sqrt{\frac{2}{3}}\right)$ equals ________

Q.47 In a production line of a factory, each packet contains four items. Past record shows that 20% of the produced items are defective. A quality manager inspects each item in a packet and approves the packet for shipment if at most one item in the packet is found to be defective. Then the probability (round off to 2 decimal places) that out of the three randomly inspected packets at least two are approved for shipment equals __________

Q.48 Let X be the number of heads obtained in a sequence of 10 independent tosses of a fair coin. The fair coin is tossed again X number of times independently, and let Y be the number of heads obtained in these X number of tosses. Then $E(X + 2Y)$ equals __________

Q.49 Let $0, 1, 0, 0, 1$ be the observed values of a random sample of size five from a discrete distribution with the probability mass function $P(X = 1) = 1 - P(X = 0) = 1 - e^{-\lambda}$, where $\lambda > 0$. The method of moments estimate (round off to 2 decimal places) of λ equals __________

Q.50 Let X_1, X_2, X_3 be a random sample from $N(\mu_1, \sigma^2)$ distribution and Y_1, Y_2, Y_3 be a random sample from $N(\mu_2, \sigma^2)$ distribution. Also, assume that (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are independent. Let the observed values of $\sum_{i=1}^{3}\left[X_i - \frac{1}{3}(X_1 + X_2 + X_3)\right]^2$ and $\sum_{i=1}^{3}\left[Y_i - \frac{1}{3}(Y_1 + Y_2 + Y_3)\right]^2$ be 1 and 5, respectively. The length (round off to 2 decimal places) of the shortest 90% confidence interval of $\mu_1 - \mu_2$ equals __________

Q. 51 – Q. 60 carry two marks each.

Q.51 $\lim_{n\to\infty}\left[n - \frac{n}{e}\left(1 + \frac{1}{n}\right)^n\right]$ equals __________

Q.52 For any real number y, let $[y]$ be the greatest integer less than or equal to y and let $\{y\} = y - [y]$. For $n = 1, 2, \ldots,$ and for $x \in \mathbb{R}$, let

$$f_{2n}(x) = \begin{cases} \left[\frac{\sin x}{x}\right], & x \neq 0, \\ 1, & x = 0, \end{cases} \quad \text{and} \quad f_{2n-1}(x) = \begin{cases} \left\{\frac{\sin x}{x}\right\}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then $\lim_{x\to 0}\sum_{k=1}^{100} f_k(x)$ equals __________

Q.53 The volume (round off to 2 decimal places) of the region in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 2$ and $y + z = 4$ equals __________

Q.54 If $ad - bc = 2$ and $ps - qr = 1$, then the determinant of $\begin{bmatrix} a & b & 0 & 0 \\ 3 & 10 & 2p & q \\ c & d & 0 & 0 \\ 2 & 7 & 2r & s \end{bmatrix}$ equals _________

Q.55 In an ethnic group, 30% of the adult male population is known to have heart disease. A test indicates high cholesterol level in 80% of adult males with heart disease. But the test also indicates high cholesterol levels in 10% of the adult males with no heart disease. Then the probability (round off to 2 decimal places), that a randomly selected adult male from this population does not have heart disease given that the test indicates high cholesterol level, equals __________

Q.56 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} ax^2, & 0 < x < 1, \\ bx^{-4}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where a and b are positive real numbers. If $E(X) = 1$, then $E(X^2)$ equals __________

Q.57 Let X and Y be jointly distributed continuous random variables, where Y is positive valued with $E(Y^2) = 6$. If the conditional distribution of X given $Y = y$ is $U(1 - y, 1 + y)$, then $Var(X)$ equals __________

Q.58 Let $X_1, X_2, \dots, X_{10}$ be i.i.d. $N(0,1)$ random variables. If $T = X_1^2 + X_2^2 + \cdots + X_{10}^2$, then $E\left(\frac{1}{T}\right)$ equals __________

Q.59 Let X_1, X_2, X_3 be a random sample from a continuous distribution with the probability density function

$$f(x) = \begin{cases} e^{-(x-\mu)}, & x > \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_{(1)} = \min\{X_1, X_2, X_3\}$ and $c > 0$ be a real number. Then $\left(X_{(1)} - c, X_{(1)}\right)$ is a 97% confidence interval for μ, if c (round off to 2-decimal places) equals __________

Q.60 Let X_1, X_2, X_3, X_4 be a random sample from a discrete distribution with the probability mass function $P(X = 0) = 1 - P(X = 1) = 1 - p$, for $0 < p < 1$. To test the hypothesis

$$H_0: p = \frac{3}{4} \quad \text{against} \quad H_1: p = \frac{4}{5},$$

consider the test:

$$\text{Reject } H_0 \text{ if } X_1 + X_2 + X_3 + X_4 > 3.$$

Let the size and power of the test be denoted by α and γ, respectively. Then $\alpha + \gamma$ (round off to 2 decimal places) equals __________

END OF THE QUESTION PAPER