Problem set 9

Spring 2018

MATHEMATICS-II (MA10002)(Vector Analysis)

1. Determine gradient and the unit normal vector for the following surfaces:
   (a) $x^2y^3 + 3xz + 2y^4 = 5$ at $(1, -2, 0)$.
   (b) $y \log x - x^2 + xz^3 = 1$ at $(1, 0, 1)$.
   (c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at (x_0, y_0, z_0).
   (d) $z = \tan(x^2 + y^2)$ at $(0, 0, 1)$.
2. Find the directional derivative of the following surfaces:
   (a) $f = e^x \cos y$ at $(0, \frac{\pi}{4})$ in the direction $\hat{\imath} + 3\hat{\jmath}$.
   (b) $f = (x^2 + y^2 + z^2)^{3/2}$ at $(-1, 1, 2)$ in the direction $\hat{\imath} - 2\hat{\jmath} + \hat{k}$.
   (c) $f = \sqrt{xy^2 + 2x^2z}$ at $(2, -2, 1)$ in the direction parallel to the z axis.
   (d) $f = 3x^4 + 2y^3 - a^2$ at $(1, 1)$ in the direction which makes an angle $30°$ to x axis, where a is some constant.
3. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ is maximum and what is its magnitude?
4. Find the angle between the two surfaces $x^2 + y^2 + z^2 = 6$ and $z = x^2 + y^2$ at the point $(1, -1, 2)$.
5. Find the constants a and b such that the surface $ax^2 - byz - (a + 2)x = 0$ will be orthogonal to the surface $4x^2y + z^3 - 4 = 0$ at the point $(1, -1, 2)$.
6. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = |\vec{r}|$, then prove that,
   (a) $\nabla r^n = nr^{n-2}\vec{r}$.
   (b) $\nabla^2 r^n = \nabla \cdot (\nabla r^n)$.
   (c) $\nabla^2 \ln r = \frac{1}{r^2}$.
7. For any constant vector $\vec{a}$ prove that

(a) $div[(\vec{a} \cdot \vec{r})\vec{r}] = 4\vec{a} \cdot \vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

(b) $\nabla \times (\vec{a} \times \vec{v}) = \vec{a}(\nabla \cdot \vec{v}) - (\vec{a} \cdot \nabla)\vec{v}$, where $\vec{v}$ is any vector field.

8. For any two scalar functions f and g prove that

   (a) $\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.

   (b) $\nabla \cdot (\nabla f \times \nabla g) = 0$.

9. If a rigid body rotates about an axis passing through the origin with angular velocity $\vec{\omega}$ and with linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that

$$\vec{\omega} = \frac{1}{2}\left(\vec{\nabla} \times \vec{v}\right)$$

10. If $\vec{F} = \frac{\vec{r}}{r^2}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then evaluate $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ and state whether $\vec{F}$ is solenoidal or irrotational.

11. If $\vec{A}$ and $\vec{B}$ are irrotational, show that $\vec{A} \times \vec{B}$ is solenoidal.

12. Show that, $r^n\vec{r}$ is solenoidal only when $n = -3$ $(r \neq 0)$.

13. Show that the vector field defined by the vector function $\vec{v} = xyz(yz\hat{i} + zx\hat{j} + xy\hat{k})$ is conservative and find the corresponding scalar potential function.

14. Suppose $\overrightarrow{F}$ = yî + (x - 2xz)ĵ - xyk̂

    Evaluate $\int\int_s(\nabla \times \overrightarrow{F})\cdot$ n ds where S is the surface of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ about the xy plane.

15. Verify Gauss theorem for $\overrightarrow{F} = x\hat{i} - y^2\hat{j} + z^2\hat{k}$ over the region bounded by $x^2 + y^2 = 4, z = 0, z = 4$.

16. Verify Stokes' theorem for $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the unit sphere and C is its boundary.

17. Use Green's theorem to evaluate the integral $\int_C[(1 + x^2)dx - x^2dy]$, where C consists the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.

18. Verify Green's theorem in the plane for $\oint(xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.