Problem Set - 7

SPRING 2018

MATHEMATICS-II (MA10002) (Integral Calculus)

1. Evaluate the integral $\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$, $(\alpha > -1)$ by applying differentiating under the integral sign.

2. Using differentiation under integral sign prove the following:
   (i) $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx = \frac{\pi}{2}\log(a+1)$, where $a \geq 0$ and $a \neq 1$,
   (ii)Using the result $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$, show that $\int_0^\infty e^{-x^2}\cos(2\alpha x)\, dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2}$,
   (iii) $\int_0^t \frac{\log(1+tx)}{1+x^2}\, dx = \frac{\tan^{-1}(t)}{2}\log(1+t^2)$

3. Let $f(x,t) = (x+t^3)^2$ then
   (i) find $\int_0^1 f(x,t)\, dx$.
   (ii) Prove that $\frac{d}{dt}\int_0^1 f(x,t)\, dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\, dx$.

4. For any real numbers x and t, let

$$f(x,t) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0, t \neq 0 \\ 0 & \text{if } x = 0, t = 0 \end{cases}$$

   and $F(t) = \int_0^1 f(x,t)\, dx$. Is $\frac{d}{dt}\int_0^1 f(x,t)\, dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\, dx$? Give the justification.

5. Find the value of the integral $\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\, dx$, where $\alpha > 0$ and deduce that
   (i) $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$
   (ii) $\int_0^\infty \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$

6. Find the value of the following integrals
   (i) $\int_0^\infty (e^{-x} - e^{-tx})\frac{dx}{x}$

(ii) $\int_0^\infty \frac{e^{-x}}{x}(a - \frac{1}{x} + \frac{1}{x}e^{-ax})\,dx$

(iii) $\int_0^1 \frac{x^a - x^b}{\log x}\,dx$

7. Find the value of $\int_0^\pi \frac{dx}{a + b\cos x}$ when $(a > 0, |b| < a)$

and deduce that $\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{\frac{3}{2}}}$.

8. Evaluate the following integrals over the region D

(i) $\int\int_D xy\,dxdy$, where D is the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = x^2/4a$.

(ii) $\int\int_D e^{x^2}\,dxdy$ where the region D is given by $R : 2y \le x \le 2$ and $0 \le y \le 1$.

(iii) $\int\int_D x^2\,dxdy$, where D is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x, y = 0$ and $x = 8$.

(iv) $\int\int_D \sqrt{xy - y^2}\,dydx$, where D is a traiangle with vertices $(0, 0), (10, 1)$ and $(1, 1)$.

9. Evaluate the following integrals by changing the order of integration

(i) $\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\,dydx$,

(ii) $\int_0^1\int_x^1 e^{y^2}\,dxdy$,

(iii) $\int_0^2\int_0^{y^2/2} \frac{y}{\sqrt{x^2 + y^2 + 1}}\,dxdy$,

(iv) $\int_0^1\int_{x^2}^{2-x} xy\,dydx$,

(v) $\int_0^\infty\int_0^x e^{-xy}y\,dydx$,