Problem Set - 2

SPRING 2018

MATHEMATICS-II (MA10002)(Linear Algebra)

1. Determine which of the following form a basis of the respective vector spaces:

   (a) $\{1+2x+x^2, 3+x^2, x+x^2\}$ of $\mathbb{P}_3$,

   (b) $\{1, \sin x, \sin^2 x, \cos^2 x\}$ of $C[-\pi, \pi]$,

   (c)
   $$\left\{ \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\}$$
   of $M_{2\times 2}$

2. Determine the basis and dimension of the following subspaces

   (a) $U = \{(x, y, z) \in \mathbb{R}^3$: x+2y+z=0 , 2x+y+3z=0 $\}$ of $\mathbb{R}^3$.

   (b) $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$: $x_2 = x_3 = x_4$ and $x_1 + x_5 = 0$ $\}$ of $\mathbb{R}^5$

   (c) Let $U = \{p \in \mathbb{P}_4 : \int_{-1}^{1} p = 0\}$.Find a basis and dimension of U.

3. Let $V = M_{2\times 2}(F)$ and
   $W_1 = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in V : a, b, c \in F \right\}$,
   $W_2 = \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in V : a, b \in F \right\}$.Find the dimension of $W_1, W_2, W_1 + W_2, W_1 \cap W_2$.

4. Check the following mappings are linear transformation or not:

   (a) T:$\mathbb{R}^3 \to \mathbb{R}^2$,defined by $T(x, y, z) = (|x|, y + z), \forall (x, y, z) \in \mathbb{R}^3$.

   (b) T:$\mathbb{P}_3 \to \mathbb{P}_4$,defined by $T(p(x)) = (x + 1)p(x) + p'(0)$.

5. Give an example of a function$\phi : \mathbb{C} \to \mathbb{C}$,such that $\phi(w + z) = \phi(w) + \phi(z), \forall w, z \in \mathbb{C}$. But ϕ is not linear.

6. Find the null space and range space of the following linear transformations.Aiso find their respective dimensions and verify the rank-nullity theorem.

   (a) $T : M_{n\times n}(F) \to F$ defined by
   T(A)=trace(A).

   (b) $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^2$,
   $T(x, y, z) = (\frac{x+y+z}{2}, \frac{x}{2})$

   (d) $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ defined by $T(A) = \dfrac{A + A^T}{2}, \forall A \in M_{2\times 2}(F)$.

7. Find the linear transformations:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2), T(2, 3) = (1, -1, 4)$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(e_1) = e_1 - e_2, T(e_2) = 2e_1 + e_3, T(e_3) = e_1 + e_2 + e_3.\{e_1, e_2, e_3\}$ is the usual basis of $\mathbb{R}^3$.

8. Find the matrix of the linear transformations w.r.t the given bases:

   (a) $D : P_3 \to P_3$ defined by $D(p(x)) = \dfrac{d^2}{dx^2}(p(x))$, w.r.t. the basis $\{1, x, x^2, x^3\}$.

   (b) $T : P_2(I\!R) \to M_{2\times 2}(I\!R)$ by
$$T(f(x)) = \begin{bmatrix} f'(0) & 2f(1) \\ 0 & f\text{'}(3) \end{bmatrix}$$
   w.r.t. the basis $\{1, x, x^2\}$ and $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

9. (a) $T : P_3(I\!R) \to P_3(I\!R)$ by $T(f(x)) = \displaystyle\int_0^x f(t)dt$. Prove that T is linear and one-one but not onto.

   (b) Prove that there does not exist a linear map $T : I\!R^5 \to I\!R^5$ such that $R(T) = N(T)$.

10. Solve the following system of equations by Gauss-elimination method:

$$(a) \quad \begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 0 \\ 3x_1 + 2x_2 - 3x_3 &= 0 \\ -4x_1 - 4x_2 + 2x_3 + x_4 &= 0 \\ 2x_1 - 4x_3 &= 0 \end{aligned} \qquad (b) \quad \begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

11. Find the rank of the matrix A using definition where
$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{bmatrix}.$$

12. Determine the rank of the following matrices by reducing to row echelon form.
$$(a) \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{bmatrix}, \qquad (b) \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{bmatrix}.$$

13. Find all x such that the rank of the matrix $\begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$ is less than 3.

14. Determine whether the following matrices are invertible or not, if it is, then compute the inverse :
$$(a) \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix}, \qquad (b) \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 5 & 4 \end{bmatrix}.$$

15. Find the value of k for which the system of equations has non-trivial solution.

$$(a) \quad \begin{aligned} x + y + z &= 0 \\ y + z &= 0 \\ ky + z &= 0 \end{aligned} \qquad (b) \quad \begin{aligned} (3k-8)x + 3y + 3z &= 0 \\ 3x + (3k-8)y + 3z &= 0 \\ 3x + 3y + (3k-8)z &= 0 \end{aligned}$$

16. If the following system

$$\begin{aligned} ax + by + cz &= 0 \\ bx + cy + az &= 0 \\ cx + ay + bz &= 0 \end{aligned}$$

has no trivial solution, then prove that $a + b + c = 0$ or $a = b = c$.

17. Solve if possible

$$\begin{aligned} x + 2y + z - 3w &= 1 \\ 2x + 4y + 3z + w &= 3 \\ 3x + 6y + 4z - 2w &= 5 \end{aligned}$$

18. Determine the condition for which the system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y - z &= b \\ 5x + 7y + az &= b^2 \end{aligned}$$

admits of (i) only one solution, (ii) no solution, (iii) many solutions.