# Tutorial Sheet - 4

**SPRING 2017**

**MATHEMATICS-II (MA10002)(Linear Algebra)** *January 2, 2017*

1. Find real numbers $x,\ y,\ z$ such that $A$ is Hermitian, where $A = \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$.

2. (a) If $(I+A)^{-1}(I-A)$ is a real orthogonal matrix, then prove that $A$ is a skew-symmetric matrix.

   (b) If $(I+A)^{-1}(I-A)$ is an unitary matrix, then prove that the matrix $A$ is a Skew-Hermitian matrix.

3. (a) If $A$ is a real orthogonal matrix and $(I+A)$ is non-singular, then prove that $(I+A)^{-1}(I-A)$ is a skew-symmetric matrix.

   (b) If $A$ is an unitary matrix and $(I+A)$ is non singular, then prove that the matrix $(I+A)^{-1}(I-A)$ is Skew-Hermitian.

4. (a) Prove that the eigenvalues of a real symmetric matrix are real.

   (b) Prove that the eigenvalues of a real skew-symmetric matrix are either purely imaginary or zero.

   (c) Show that all eigenvalues of a Hermitian matrix are real.

   (d) Show that the eigenvalues of a real skew-hermitian matrix are either purely imaginary or zero.

5. For each of the following matrices, find all the eigenvalues and corresponding eigenvectors.

   (i) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

   (iv) $H = \begin{pmatrix} -1 & 0 & -2i \\ 0 & 2 & 0 \\ 2i & 0 & -1 \end{pmatrix}$ (v) $S = \begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}$.

6. Prove that the eigenvalues of an unitary matrix are of modulus 1.

7. If $\lambda$ is an eigenvalue of a Skew-Hermitian matrix, then prove that $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$.

8. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find its inverse. Also, find $\alpha, \beta \in \mathbb{R}$ such that the expression $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ is of the form $\alpha A + \beta I$.

9. Verify the Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$. Using this find $A^{-1}$, if exists.

10. Examine whether $A$ is similar to $B$, where

(a) $A = \begin{pmatrix} 5 & 5 \\ -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

11. (a) If $A$ and $B$ are similar, then prove that $A$ and $B$ have the same characteristic polynomial.

    (b) Is the converse true?

12. Which of the following matrices $A$

(i) $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

is diagonalizable. If so, obtain an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.

13. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then find $A^{100}$ using Cayley-Hamilton theorem.

14. (a) The eigenvectors of a $3 \times 3$ matrix $A$ corresponding to the eigenvalues $1, 2, 3$ are $[1, 2, 1]^T$, $[2, 3, 4]^T$ and $[1, 4, 9]^T$ respectively. Find the matrix $A$, and hence find $A^{500}$

    (b) Let $A$ be a $3 \times 3$ real matrix having the eigenvalues $2, 3, 1$. Let $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the eigen values of $A$ corresponding to the eigenvalues $2, 3, 1$ respectively. Then find the matrix $A$. And hence find $A^n$, for any $n \in \mathbb{N}$.