Tutorial Sheet - 2

SPRING 2017

MATHEMATICS-II (MA10002)(Linear Algebra) *January 9, 2017*

1. Determine which of the following forms a basis of the respective vector spaces:
   (a) $\{(1,5,-6),\ (2,1,8),\ (3,-1,4)\}$ of $\mathbb{R}^3$,
   (b) $\{1, x-2, (x-2)^2, (x-2)^3\}$ of $\mathbb{P}_3$,
   (c) $\{1, \sin x, \sin^2 x, \cos^2 x\}$ of $C[-\pi,\pi]$,
   (d) $\left\{\begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\right\}$ of $M_{2\times 2}$.

2. Let U be the subspace of $\mathbb{C}^5$ defined by
   $U = \{(z_1,\ z_2,\ z_3,\ z_4,\ z_5) \in \mathbb{C}^5\ :\ 6z_1 = z_2,\ z_3 + 2z_4 + 3z_5 = 0\}$. Find a basis of U.

3. Determine the basis and dimension of the following subspaces
   (a) $U = \{(x,y,z) \in \mathbb{R}^3\ :\ x + 2y + z = 0,\ 2x + y + 3z = 0\}$ of $\mathbb{R}^3$,
   (b) $U = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5\ :\ x_1 + x_2 + x_3 = 0,\ 3x_1 - x_4 + 7x_5 = 0\}$ of $\mathbb{R}^5$.

4. (a) Let $U = \{p \in \mathbb{P}_3\ :\ p(1) = 0\}$ and $W = \{p \in \mathbb{P}_3\ :\ p'(1) = 0\}$. Then find $dim(U \bigcap W)$ and $dim(U + W)$.
   (b) Let $U = \{p \in \mathbb{P}_4\ :\ \int_{-1}^{1} p = 0\}$
   (i) Find a basis and dimension of U,
   (ii) Extend the basis in part (a) to a basis of $\mathbb{P}_4$.

5. Check the following mappings are linear transformation or not:
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x,y,z) = (x+2y+3z, 3x+2y+z, x+y+z),\ \forall (x,y,z) \in \mathbb{R}^3$,
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x,y,z) = (|x|, y+z),\ \forall (x,y,z) \in \mathbb{R}^3$,
   (c) $T : \mathbb{P}_3 \to \mathbb{P}_4$ defined by $T(p(x)) = xp(x) + p(1),\ \forall p(x) \in \mathbb{P}_3$,
   (d) $T : M_{2\times 2} \to M_{2\times 2}$ defined by $T(A) = \dfrac{A + A^T}{2},\ \forall A \in M_{2\times 2}$.

6. Give an example of a function $\phi : \mathbb{C} \to \mathbb{C}$, such that $\phi(w+z) = \phi(w) + \phi(z)\, \forall\, w, z \in \mathbb{C}$. But ϕ is not linear. (Here $\mathbb{C}$ is a vector space over $\mathbb{C}$).

7. Find the null space and range space of the following linear transformations. Also find their respective dimensions and verify the rank-nullity theorem:
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x,y,z) = (x+y+z, 2x+y+2z, x+2y+z),\ \forall (x,y,z) \in \mathbb{R}^3$
   (b) $T : M_{2\times 2} \to M_{2\times 2}$ defined by $T(A) = \dfrac{A + A^T}{2},\ \forall A \in M_{2\times 2}$
   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(x,y) = (\dfrac{x+y}{2}, \dfrac{x+y}{2}),\ \forall (x,y) \in \mathbb{R}^2$.
   (d) $T : \mathbb{R}^3 \to \mathbb{R}$, defined by $T(x,y,z) = x+y+z,\ \forall (x,y,z) \in \mathbb{R}^3$.

8. Find the linear transformations :
(a) $T : \mathbb{R}^3 \to \mathbb{R}$ where $T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0,0,1) = -2$.
(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(e_1) = e_1 - e_2$, $T(e_2) = 2e_1 + e_3$, $T(e_3) = e_1 + e_2 + e_3$. $\{e_1, e_2, e_3\}$ is the usual basis of $\mathbb{R}^3$.
(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(2,1,1) = (1,1,1)$, $T(1,2,1) = (1,1,1)$, $T(1,1,2) = (1,1,1)$.

9. Find the matrix of the linear transformations w.r.t. the given bases:
(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x,y,z) = (x+y+z, x+z, x+y)$ $(x,y,z) \in \mathbb{R}^3$: with respect to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$.
(b) $D : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $D(p(x)) = \dfrac{d^2}{dx^2}(p(x))$, w.r.t. the basis $\{1, x, x^2, x^3\}$,
(c) $T : \mathbb{P}_3 \to \mathbb{P}_4$ defined by $T(p(x)) = (2+x)p(x)$, w.r.t. the basis $\{1, x, x^2, x^3\}$ and $\{1, x, x^2, x^3, x^4\}$ respectively,

10. (i) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^2$ such that $N(T) = \{(x,y,z,w) \in \mathbb{R}^4 : x = 5y, z = 7w\}$. Prove that T is surjective.
(ii) U is a 3-dimensional subspace of $\mathbb{R}^8$ and $T : \mathbb{R}^8 \to \mathbb{R}^5$ is a linear map such that $N(T) = U$. Prove that T is surjective.
(iii) Prove that there does exist not a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $R(T) = N(T)$.
(iv) Prove that there does not exist a linear map from $\mathbb{R}^5$ to $\mathbb{R}^2$ where null space is $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = x_4 = x_5\}$.