MA20103 Partial Differential Equations
Assignment 5

1. Determine the region in which the given PDE is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form

   (a) $xu_{xx} + u_{yy} = x^2$

   (b) $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$

   (c) $e^x u_{xx} + e^y u_{yy} = u$

   Answers:

   (a) $x < 0$, hyperbolic with $u_{\xi\eta} = \frac{1}{4}\frac{\xi-\eta}{4}^4 - \frac{1}{2}\frac{1}{\xi-\eta}(u_\xi - u_\eta)$,
   $x = 0$, parabolic; and the given equation would then be in canonical form already.
   $x > 0$, elliptic and the canonical form is $u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta + \frac{\beta^4}{16}$.

   (b) Parabolic everywhere with $u_{\eta\eta} = \frac{2\xi}{\eta^2}u_\xi + \frac{1}{\eta^2}e^{\frac{\xi}{\eta}}$.

   (c) Elliptic everywhere for finite values of x and y with

   $$u_{\alpha\alpha} + u_{\beta\beta} = u - \frac{1}{\alpha}u_\alpha - \frac{1}{\beta}u_\beta$$

2. Find the characteristics and reduce the following equations to canonical form:

   (a) $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$.
   Answer: $u_{\xi\eta} = 0$

   (b) $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$.
   Answer: $u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2}u_\alpha - 2\sqrt{2}u_\beta - \frac{1}{2}u + \frac{1}{2}\exp(\frac{\beta}{\sqrt{2}})$

   (c) $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$
   Answer: $u_{\xi\eta} = \frac{7}{9}(u_\xi + u_\eta) - \frac{1}{9}\sin\left[\frac{(\xi-\eta)}{3}\right]$

   (d) $x^2u_{xx} - y^2u_{yy} - u_x = 1 + 2y^2$
   Answer: Hyperbolic with $u_{\xi\eta} = \frac{1}{2}(1 + \frac{1}{2}\sqrt{\frac{\eta}{\xi}})u_\eta - \frac{1}{4}\frac{1}{\sqrt{\xi\eta}} - \frac{1}{4\xi\eta} - \frac{1}{2}$

   (e) $u_{xx} + yu_{yy} + \frac{1}{2}u_y + 4yu_x = 0$
   Answer: $y > 0$, Elliptic with $u_{\alpha\alpha} + u_{\beta\beta} = \alpha^2 u_\beta$
   $y = 0$, Parabolic with $u_{xx} + \frac{1}{2}u_y = 0$
   $y < 0$, Hyperbolic with $u_\xi\eta = \frac{1}{16}(\xi + \eta)^2(u_\eta - u_\xi)$

   (f) $y^2u_{xx} - x^2u_{yy} = 0$
   Answer: $2(\xi^2 - \eta^2)u_{\xi\eta} - \eta u_\xi + \xi u_\eta = 0$ except when $x = 0$ or $y = 0$.

3. Classify each of the following equations and reduce it to canonical form

   (a) $u_{xx} - (sech^4 x)u_{yy} = 0$
   Answer: $u_{\xi\eta} = [4 - (\xi - \eta)^2]^{-1}(\eta - \xi)(u_\xi - u_\eta)$ in the domain $(\eta - \xi)^2 < 4$

   (b) $y^2u_{xx} + 2xyu_{xy} + 2x^2u_{yy} + xu_x = 0$
   Answer: $u_{\alpha\alpha} + u_{\beta\beta} = [2\beta(\alpha + \beta)]^{-1}[\alpha u_\alpha - (\alpha + 2\beta)u_\beta$

4. Change the PDE

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$$

into canonical form and integrate to obtain general solution.
Answer: $\eta u_{\xi\eta} + u_\xi = 0, \quad u(x,y) = f(y-x) + \frac{1}{y-x}g(y^2 - x^2)$ where f and g are arbitrary functions.

5. Classify the second order PDE

$$\frac{3}{4}u_{xx} - 2yu_{xy} + y^2u_{yy} + \frac{1}{2}u_x = 0$$

depending on the domain. Reduce it to canonical form and integrate to obtain the general solution.
Answer: $2u_{\xi\eta} + u_\eta = 0$, for $y \neq 0$; $\frac{3}{2}u_{xx} + u_x = 0$, for $y = 0$.

6. Find the general solution of the following PDEs:

   (a) $yu_{xx} + 3yu_{xy} + 3u_x = 0; \quad y \neq 0$

   (b) $u_{xx} - 2u_{xy} + u_{yy} = 135 sin(3x + 2y)$

   (c) $3u_{xx} - 2u_{xy} - 5u_{yy} = 3x + y + e^{x-y}$

****Good Luck****